Equivalence and transition diagrams for graphs on surfaces by local transformations

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Preface

This thesis is written on the subject “Equivalence and transition diagrams for the graphs on surfaces by local transformations”, and it is to be submitted to the degree of Doctor of Science at Yokohama National University.

When I was an undergraduate school student in Konan University, I first knew Graph Theory in a class. At that time, Professor Haruko Okamura had the class, and I felt that the learning is very interesting. Then, I told her that I wanted to go to the graduate school in Konan University to study Graph Theory. However, in that year, since Okamura would retire from the position in the university, I was at a loss what to do. Such my looks disposed her to introduce graduate schools in other universities. One of them is Yokohama National University, that is, she suggested me to go to Professor Atsuhiro Nakamoto’s seminar. Moreover, she had taught me basic things for Graph Theory on an individual basis until I entered the graduate school in Yokohama National University.

Then, I began to study Graph Theory in Yokohama National University, but the graph theory I met was slightly different from the pure graph theory, since Nakamoto was working in Topological Graph Theory. Therefore, I naturally began to study graphs embedded on surfaces.

In the master’s course, I fortunately met a very good problem while preparing the master’s thesis, which is about “Transformations in hexangulations on the sphere”. When I got the problem, I thought, at first, that it was just an exercise for me. However, as my research progressed, the interesting phenomenon was getting. I first proved that any two hexangulations on the sphere with the same number of vertices can be transformed into each other, up to homeomorphism, by a sequence of three specified transformations A, B and C. Then, repeating many attempts and failures, I could, at last, completed the transition diagram for hexangulations on the sphere by the three transformations A, B and C. In the diagram, we can exactly see when each of transformations is needed, and this fact does not appear in researches on triangulations and quadrangulations. Moreover, by this success, I also completed the transition diagram for pentangulations on the sphere by two specified transformations, which has an interesting structure similar to that for hexangulations on the sphere. In this way, I solved several problems on equivalence and transition diagrams for N-angulations on closed surfaces.

In this thesis, I will present my work on Topological Graph Theory, especially Equivalence and transition diagrams for graphs on surfaces by local transformations, done during three years and nine months in the period when I was in the master’s and the doctor’s courses. This thesis consists of several chapters as follows:

At first, we roughly introduce our research and results on diagonal transformations. Then, in Chapter 1, we prepare several basic terminologies and notations on Graph Theory
and Topological Graph Theory for explaining our research. In Chapter 2, we describe equivalence for $N$-angulations by diagonal transformations. Following this, we focus on the structure of transition diagrams for $N$-angulations by diagonal transformations in Chapter 3, which is our main topic. Moreover, in Chapter 4, we consider the diameter of those transition diagrams. In the final chapter, we describe diagonal transformations in $N$-angulations with specified properties and local transformations other than diagonal transformations.

Finally, my deepest appreciation goes to Professor Atsuhiro Nakamoto who is my supervisor. Thanks to his thorough and enthusiastic coaching, I was able to study mathematics and could grow up mentally. Then, I owe a very important debt to Professor Seiya Negami whose opinions and information have helped me very much throughout the production of this study, and I wish to express my gratitude to Professor Haruko Okamura who gave me a good opportunity to go to the master’s course. I would also like to express my gratitude to my family and members of the laboratory, where I belong, for their moral support and warm encouragements. Moreover, I would like to express my gratitude to Japan Society for the Promotion of Science for their financial support.

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Papers underlying on the thesis


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Contents

Preface 1

Papers underlying the thesis 3

Contents 4

Introduction 6

1 Foundations 14
  1.1 Graphs .................................................. 14
  1.2 Planar graphs .......................................... 18
  1.3 Embeddings ............................................ 22

2 Equivalence for $N$-angulations by diagonal transformations 25
  2.1 Definitions ............................................ 25
  2.2 Triangulations .......................................... 26
    2.2.1 Classical results .................................... 26
    2.2.2 General case ....................................... 29
  2.3 Quadrangulations ........................................ 32
    2.3.1 Diagonal slides and diagonal rotations ............ 32
    2.3.2 Bipartite case ..................................... 34
    2.3.3 Non-bipartite case ................................ 35
  2.4 $N$-angulations for $N \geq 5$ ............................ 36
    2.4.1 Necessity of the transformations .................. 37
    2.4.2 Proof of Theorem 2.21 .............................. 38
    2.4.3 Proof of Theorem 2.22 .............................. 44
    2.4.4 Extension of the results ........................... 49

3 Structure of transition diagrams 51
  3.1 Triangulations and quadrangulations .................... 51
  3.2 Pentangulations ......................................... 51
    3.2.1 The structure $\mathcal{X}$ in pentangulations ....... 53
    3.2.2 Proof of Theorem 3.1 ............................... 54
  3.3 Hexangulations ......................................... 59
    3.3.1 1-subdivided triangulations ....................... 61
    3.3.2 Hexangulations in $\mathcal{H}_{m,n}$ with $m \neq 3n - 6$ (or $n \neq 3m - 6$) ... 63
3.3.3 Problems on diagonal transformations in hexangulations . . . . . . 75
3.4 Note on $N$-angulations for $N \geq 7$ . . . . . . . . . . . . . . . . . . . . . . . 75

4 Diameter of transition diagrams 77
4.1 Known results on triangulations . . . . . . . . . . . . . . . . . . . . . . . . 77
4.2 Results on quadrangulations . . . . . . . . . . . . . . . . . . . . . . . . . . 80
4.3 Proof of Theorem 4.9 and Corollary 4.10 . . . . . . . . . . . . . . . . . . . 82
4.4 The number of diagonal transformations in $N$-angulations for $N \geq 5$ . . 87

5 A survey of local transformations 88
5.1 Diagonal transformations preserving specified properties . . . . . . . . . 88
  5.1.1 Minimum degree conditions . . . . . . . . . . . . . . . . . . . . . . . 88
  5.1.2 Labeled triangulations . . . . . . . . . . . . . . . . . . . . . . . . . . 90
  5.1.3 Pseudo-triangulations . . . . . . . . . . . . . . . . . . . . . . . . . . 91
5.2 Other transformations in graphs on surfaces . . . . . . . . . . . . . . . . . 92
  5.2.1 $N$-flips and $P_2$-flips in even triangulations . . . . . . . . . . . . . 92
  5.2.2 Signed diagonal flips . . . . . . . . . . . . . . . . . . . . . . . . . . . 95
  5.2.3 Simultaneous flips in triangulations . . . . . . . . . . . . . . . . . . . 96
  5.2.4 Edge rotations in planar graphs . . . . . . . . . . . . . . . . . . . . . 97
  5.2.5 Proof of Theorem 5.18 . . . . . . . . . . . . . . . . . . . . . . . . . . 100
5.3 Flips in geometric setting . . . . . . . . . . . . . . . . . . . . . . . . . . . . 103
  5.3.1 Definitions and general results . . . . . . . . . . . . . . . . . . . . . . 104
  5.3.2 Delaunay flips . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 105
  5.3.3 $k$-triangulations . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 106
5.4 Edge operations in abstract graphs . . . . . . . . . . . . . . . . . . . . . . 107
  5.4.1 Distance between graphs by edge operations . . . . . . . . . . . . . . 107
  5.4.2 Distance graphs . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 109
  5.4.3 Equivalence for graphs with the same degree sequence . . . . . . . . . 109

Bibliography 111

Index 116
Introduction

A graph consists of finitely many points, called vertices, and arcs, called edges, each of which joins a pair of vertices. For example, a graph $G$ stands for some relation among a finite set $X$, where the vertex set of $G$, denoted by $V(G)$, corresponds to the elements of $X$ and the edge set of $G$, denoted by $E(G)$, stands for a relation between a pair of elements of $X$. A graph can be regarded as a mathematical model which expresses such structures of finite sets with some relation.

When we deal with a problem of graph theory, we often draw a graph on a paper. If we consider only the combinatorial structure of the graph, then we don’t care about crossings of edges of the graph drawn on the paper since a crossing of edges has no meaning. However, we would like to find a great meaning of the crossing of edges when we draw a graph on the plane or other compact 2-dimensional manifolds. In this thesis, attaching an importance to this point of view, we shall mainly deal with graphs drawn on surfaces without crossing edges.

When we discuss embeddings of graphs into surfaces, we regard graphs as 1-dimensional topological spaces, not only as combinatorial objects. Roughly speaking, to embed a graph into a surface $F^2$ is to draw the graph on $F^2$ without crossing edges. It is sometimes effective to regard an embedding as an injective continuous map $f: G \to F^2$. We deal with $G$ and $f(G)$ as the same object intuitively. However, to distinguish $G$ from the embedded one $f(G)$, we sometimes call $G$ an abstract graph while we call $f(G)$ an embedding. In this thesis, we often denote an embedded graph by $G$.

Now, if $G$ is embedded in a closed surface $F^2$, then $G$ can be regarded as a subset of $F^2$. Each component of $F^2 - G$ is called a face of $G$ embedded in $F^2$. A closed walk $W$ (resp., cycle $C$) of $G$ which bounds a face $F$ of $G$ is called the boundary walk (resp., boundary cycle) of $F$. An embedded graph $G$ is said to be a 2-cell embedding, or $G$ is said to be 2-cell embedded in $F^2$ if each face of $G$ is homeomorphic to an open 2-cell, that is, $\{(x, y) \in R^2 \mid x^2 + y^2 < 1\}$. After this, we simply call 2-cell embeddings embeddings. For a graph $G$ embedded on a closed surface $F^2$, we denote the face set of $G$ by $F(G)$, and denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. Moreover, for any face (or a 2-cell region) $f$ in a graph $G$ on a surface, $\partial f$ denotes the boundary walk of $f$.

Let $G_1$ and $G_2$ be two graphs embedded on closed surfaces $F^2_1$ and $F^2_2$, respectively. Two graphs $G_1$ and $G_2$ are said to be homeomorphic to each other if there exists a homeomorphism $h: F^2_1 \to F^2_2$ with $h(G_1) = G_2$ which induces an isomorphism from $G_1$ to $G_2$. In this case, we also say that $G_1 \subset F^2_1$ and $G_2 \subset F^2_2$ are the same ones up to homeomorphism.

An $N$-angulation $G$ on a closed surface $F^2$ is an embedding of a 2-connected simple
graph on $F^2$ such that every face of $G$ is bounded by a cycle of length $N$, where $N \geq 3$ is an integer. In particular, if $N = 3, 4, 5$ or 6, then the corresponding graph is called a triangulation, a quadrangulation, a pentangulation or a hexangulation, respectively. Note that if $N$ is even, then every $N$-angulation on the sphere is always bipartite. (Since every bipartite graph $G$ is 2-colorable, $V(G)$ can be colored by two colors, say black and white, and the sets denoted by $V_B(G)$ and $V_W(G)$, respectively. The bipartition size of $G$ means that $|V_B(G)|, |V_W(G)|$, where for any set $X$, the cardinality of $X$ is denoted by $|X|$.) Moreover, we sometimes call an $N$-angulation on a closed surface $F^2$ an $N$-angulation if we do not need to specify $F^2$.

In an $N$-angulation, let $P=x_1y_1y_2\cdots y_{l-2}y_{l-1}x_k$ be a path of length $l$ ($1 \leq l \leq \lfloor \frac{N}{2} \rfloor, l + k = N + 1$), which is shared by two faces $F_1$ and $F_2$, where $\partial F_1 = x_1x_2\cdots x_{k-1}y_{l-1}\cdots y_2y_1$ and $\partial F_2 = x_1x_{2k-2}x_{2k-3}\cdots x_{k+1}y_{l-1}\cdots y_2y_1$, respectively. Replacing a path $P$ with a path $P' = x_2y_1y_2\cdots y_{l-2}y_{l-1}x_{k+1}$ is called a diagonal transformation (see Figure 1).

![Figure 1: A diagonal transformation in an $N$-angulation](image1)

So, for $N$-angulations, there are $\lfloor \frac{N}{2} \rfloor$ kinds of diagonal transformations, depending on the length $l$ of the path $P$. When this transformation breaks the simpleness or 2-connectedness of graphs, we don’t apply it. Two $N$-angulations are said to be equivalent if they can be transformed into each other by diagonal transformations, up to homeomorphism.

In the literature, Wagner [78] proved that any two triangulations on the sphere with the same number of vertices are equivalent. By the definition of diagonal transformations, flipping an edge shown in Figure 2 is a unique diagonal transformation in triangulations, which is called a diagonal flip.

![Figure 2: A diagonal flip](image2)

For the torus, the projective plane and the Klein Bottle, Dewdney [20], Negami and
Watanabe [68] proved the same facts. It seemed to be difficult to extend the theorem for general closed surfaces since these theorems strongly depend on the topology of individual surfaces. However, Negami found a good breakthrough for the extension and proved the following theorem.

**Theorem 0.1 (Negami [61])** For any closed surface $F^2$, there exists a positive integer $N(F^2)$ such that any two triangulations $G$ and $G'$ on $F^2$ are equivalent if $|V(G)| = |V(G')| \geq N(F^2)$.

For quadrangulations, Nakamoto [53] proved the following. By the definition, sliding an edge shown in the left hand of Figure 3 and rotating a path of length 2 shown in the right hand of Figure 3 are diagonal transformations in quadrangulations, which are called a diagonal slide and a diagonal rotation, respectively. Note that a diagonal slide preserves the bipartition size but a diagonal rotation does not.

**Theorem 0.2 (Nakamoto [53])** For any closed surface $F^2$, there exists an integer $M(F^2)$ such that any two bipartite quadrangulations on $F^2$ with $N \geq M(F^2)$ vertices are equivalent.

Observe that every quadrangulation on the sphere is bipartite, but for any other closed surface $F^2$, there exists a non-bipartite quadrangulation on $F^2$. However, Nakamoto [52] also extended the theorem to the non-bipartite case for any non-spherical closed surface, where the equivalence can be described by a notion called a “cycle parity”.

![Figure 3: A diagonal slide and a diagonal rotation](image)

Similarly, for pentangulations and hexangulations on the sphere, we proved the following theorems. By the definition of diagonal transformations, those for pentangulations (resp., hexangulations) are the transformations $A$ and $B$ shown in Figure 4 (resp., the transformations $A$, $B$, and $C$ shown in Figure 5). Note that Theorem 0.3 is a corollary of the main result in [36].

**Theorem 0.3 (Matsumoto et al. [36])** Any two pentangulations on the sphere with the same number of vertices are equivalent.

**Theorem 0.4 (Matsumoto [45])** Any two hexangulations on the sphere with the same number of vertices are equivalent.

In the above theorems, we cannot omit each of transformations from those statements. The transformation $A$ is needed since there exist infinitely many pentangulations on the sphere with the minimum degree at least 3 (for example, the dodecahedron is one of them,
see Figure 6), and the standard form shown in Figure 7 of pentangulations on the sphere requires only $B$, which consists of paths of length 2 and 3 with the middle vertices of degree exactly 2 with the same ends $u$ and $v$.

For each of diagonal transformations in hexangulations, there exists a hexangulation on the sphere requiring it as follows (cf. [45]):

The standard form of hexangulations on the sphere consists of paths of length 3 in which each middle vertex has degree exactly 2 (see Figure 8). Clearly, only $C$ can be applied to the graph. A 1-subdivided triangulation is a hexangulation on the sphere obtained from a plane triangulation $T$ by subdividing each edge of $T$ with a single vertex of degree 2 (see Figure 9) to which neither $A$ nor $C$ can be applied. There is a hexangulation obtained from a plane quadrangulation by adding a path of length four into each face as a diagonal (see Figure 10) to which neither $B$ nor $C$ can be applied.

In [45], it was conjectured that for any positive integer $N \geq 7$, any two $N$-angulations on the sphere with the same number of vertices are equivalent. However, it seemed that the proof would be a routine with a case-by-case argument. Then, we develop a more general technique for proving the statement, and hence, we have the following.

**Theorem 0.5 (Matsumoto)** For any fixed integer $N \geq 7$, any two $N$-angulations on the sphere with the same number of vertices are equivalent.

By the above theorems, for any integer $N \geq 3$, any two $N$-angulations on the sphere
with the same number of vertices are equivalent. In Chapter 2, we introduce some known results and show our results on the equivalence for $N$-angulations on the sphere.

In Chapter 3, we focus on the transition diagram for $N$-angulations by diagonal transformations. For any positive integer $N \geq 3$, a transition diagram for $N$-angulations by diagonal transformations is the graph in which each vertex represents an $N$-angulation with some number of vertices, and two vertices are adjacent in the diagram if and only if they are transformed into each other by a single diagonal transformation.

We consider the structure of the transition diagram for hexangulations on the sphere, which is a main topic in this thesis. Let $\mathcal{D}_N$ be the transition diagram of hexangulations on the sphere with $N$ vertices by diagonal transformations, and let $\mathcal{D}_{m,n}$ be the subset of $\mathcal{D}_N$ consisting of hexangulations on the sphere with $m$ black vertices and $n$ white ones (note that any hexangulation on the sphere is bipartite). Theorem 0.4 asserts that $\mathcal{D}_N$ is connected. Since only $B$ among $A$, $B$ and $C$ changes the bipartition size of hexangulations, $B$ is necessary to transform an element in $\mathcal{D}_{m,n}$ into another in $\mathcal{D}_{m+k,n-k}$ for any integer $k \neq 0$.

First, we consider a question: Is there a pair $(m,n)$ such that $\mathcal{D}_{m,n} = \emptyset$? Observe that every 1-subdivided triangulation has $3n - 6$ black (resp., white) vertices and $n$ white (resp., black) vertices since every triangulation on the sphere with $n$ vertices has exactly $3n - 6$ edges by Euler’s formula (this formula means $V - E + F = 2$, where $V$, $E$ and $F$ are the number of vertices, edges and faces, respectively). In fact, this number bounds the balance of the number of black vertices and white ones as follows:

**Lemma 0.6** If $m \geq n$, then $\mathcal{D}_{m,n}$ with $m > 3n - 6$ is empty. Moreover, if $m = 3n - 6$, then each element in $\mathcal{D}_{m,n}$ is a 1-subdivided triangulation.

The above lemma answers the previous question, that is, $\mathcal{D}_{m,n} = \emptyset$ for every pair $(m,n)$ such that $m > 3n - 6$ or $n > 3m - 6$.

Next, focusing on $\mathcal{D}_{m,n}$ with $m \neq 3n - 6$ (it suffices to consider $m \geq n$ by symmetry), we consider whether any two elements in $\mathcal{D}_{m,n}$ can be transformed into each other only by $A$ and $C$. Note that there are hexangulations on the sphere requiring only $A$ and $C$ shown in Figures 10 and 8. However, though there are infinitely many hexangulations on the sphere requiring $A$, we cannot find hexangulations requiring $C$ which are not isomorphic to the standard form. Hence, we conjectured that there are only few graphs requiring $C$. 

![Figure 8: The standard form](image1.png)

![Figure 9: A 1-subdivided triangulation](image2.png)

![Figure 10: A hexangulation requiring $A$](image3.png)
but we prove the following theorem. Surprisingly, we do not need $C$ to transform any element in $D_{m,n}$ with $m \neq 3n - 6$ and $m \neq n$ into another in the same set. Moreover, we can see that $C$ is needed at most once to transform the standard form into any other.

**Theorem 0.7** Let $D_{m,n}$ be the set of all hexangulations with the bipartition size $(m, n)$, where $m \geq n$ and $m \neq 3n - 6$. Then

(i) if $m \neq n$, any two hexangulations in $D_{m,n}$ can be transformed into each other only by $A$, and

(ii) if $m = n$, any two hexangulations in $D_{m,n}\\{S\}$ can be transformed into each other only by $A$, where $S$ is the standard form.

Finally, we consider $D_{3n-6,n}$ and $D_{m,3m-6}$. By the definition of 1-subdivided triangulations, for each element in $D_{3n-6,n}$ (resp., $D_{m,3m-6}$), every black (resp., white) vertex has degree exactly 2. Hence, only $B$ can be applied to each element in $D_{3n-6,n}$ (or $D_{m,3m-6}$), and $D_{3n-6,n}$ and $D_{m,3m-6}$ consist of isolated vertices.

By summarizing the above results and observations, we can complete the transition diagram of hexangulations on the sphere which has a very interesting structure (see Figure 11).

Moreover, we can also make the transition diagram $P_N$ of pentangulations on the sphere which has an interesting structure as shown in Figure 12. By the diagram, we can see that almost pentangulations with the same number of vertices can be transformed into each other only by $A$. For the details, we explain them in Chapter 3.

In Chapter 4, we focus on the diameter of transition diagrams. Since the distance in transition diagrams corresponds to the minimum number of diagonal transformations which needs to transform an $N$-angulation into another, the diameter is the maximum number of the distance of any two elements in the transition diagram. In the proof of results for the equivalence, the number of diagonal transformations can easily be estimated at $O(n^2)$, where $n$ is the number of vertices. However, it seems that the number is not
best possible, and hence, the study for improving the number of diagonal transformations has begun.

Firstly, Komuro [38] improved the algorithm of the proof of Wagner’s result [78], and then, the same fact was proved by using at most $8n - 48$ diagonal flips, where $n$ is the number of vertices. Moreover, Komuro [38] also proved that the order of the number of diagonal flips is best possible, that is, he constructed two triangulations $T$ and $T'$ on the sphere with $n$ vertices which require $\Omega(n)$ diagonal flips to transform $T$ into $T'$. After this, by a clever idea, Mori et al. [50] improved Komuro’s result as follows.

**Theorem 0.8 (Mori et al. [50])** Any two triangulations on the sphere with $n \geq 6$ vertices can be transformed into each other by at most $6n - 30$ diagonal flips.

Afterward, for the projective plane [48], the torus [73] and any other surface [49], the corresponding results are re-proved by $O(n)$ diagonal flips, where $n$ is the number of vertices.

We focus on the number of diagonal transformations in quadrangulations. Recently, Nakamoto and Suzuki [58] proved that any two quadrangulations on the sphere with $n$ vertices can be transformed into each other by $6n - 32$ diagonal transformations. However, their proof does not preserve the bipartition size $(B; W)$ of a given quadrangulation. On the other hand, Nakamoto [53] proved that there exist two positive integers $M(F^2)$ and $N(F^2)$ such that any two bipartite quadrangulations on a closed surface $F^2$ with the same bipartition size $(B; W)$ can be transformed into each other by $O(BW)$ diagonal slides, where $B \geq M(F^2)$ and $W \geq N(F^2)$. Therefore, we focus on quadrangulations on the sphere with the same bipartition size, and re-prove the above Nakamoto’s theorem [53] in the sphere case, as follows.

**Theorem 0.9 (Matsumoto and Nakamoto [47])** Any two quadrangulations $G$ and $G'$ on the sphere with $|V_B(G)| = |V_B(G')| \geq |V_W(G)| = |V_W(G')| \geq 3$ can be transformed into each other only by at most $10|V_B(G)| + 16|V_W(G)| - 64$ diagonal slides.

By summarizing the results described above, we have Table 1 for $N$-angulations on the sphere. For equivalence for $N$-angulations on the sphere, Wagner [78], Nakamoto [53] and we [36, 45] proved for $N = 3, 4$ and $5, 6$, respectively. For $N \geq 7$, we also have a
similar result for equivalence. For triangulations and quadrangulations (cf. [78] and [53]), we do not find a very interesting phenomenon for the transition diagrams. On the other hand, we can make the transition diagram of hexangulations on the sphere which has a very interesting structure. Moreover, we can also make the transition diagram of pentangulations on the sphere with a similar substructure to the hexangulation case. For the diameter of transition diagrams, there are some known results only for triangulations and quadrangulations on the sphere. Then, for quadrangulations with the same bipartition size, we can estimate the number of diagonal slides at a linear order depending on the number of vertices. Note that there is no result in each part with “?” in the table.

<table>
<thead>
<tr>
<th>$N$</th>
<th>Equivalence</th>
<th>Transition diagram</th>
<th>Diameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$\circ$ (Wagner [78])</td>
<td>$\triangle$ ([78])</td>
<td>$6n - 30$ (Mori et al. [50])</td>
</tr>
<tr>
<td>4</td>
<td>$\circ$ (Nakamoto [53])</td>
<td>$\triangle$ ([53])</td>
<td>$6n - 32$ (Nakamoto and Suzuki [58])</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>$\circ$ (M. et al. [36])</td>
<td>$\circ$ ([36])</td>
<td>$10B + 16W - 64$ (M. and Nakamoto [47])</td>
</tr>
<tr>
<td>6</td>
<td>$\circ$ (M. [45])</td>
<td>$\circ$ (M. [46])</td>
<td>$?$</td>
</tr>
</tbody>
</table>

Table 1: Results for equivalence, the transition diagram and the diameter for $N$-angulations on the sphere

In this thesis, we first prepare fundamental terminologies in Chapter 1. Next, we describe the results of the first (equivalence), second (transition diagrams) and third (diameter) lines on Table 1 in Chapters 2, 3 and 4, respectively. Finally, we introduce results for local transformations other than diagonal transformations and several open problems in Chapter 5.
Chapter 1

Foundations

In this chapter, we shall present basic terminologies and notations which will be needed in this thesis.

1.1 Graphs

A graph $G$ is a pair of sets, $V(G)$ and $E(G)$, where $V(G)$ is nonempty and $E(G)$ is a set of 2-element subsets of $V(G)$. For a set $X$, let $|X|$ denote the cardinality of $X$. The elements of $V(G)$ are called vertices (the singular is called a vertex) of $G$ and the elements of $E(G)$ are called edges of $G$. For an edge $e = uv \in E(G)$, the vertices $u$ and $v$ are called endvertices of $e$.

![Figure 1.1: A graph](image)

We say that an edge $e = xy$ is incident with its endvertices, and it joins its endvertices. In this case, $x$ and $y$ are said to be adjacent vertices of $G$. If $x = y$, then the edge is called a loop, and if at least two edges join $x$ and $y$, then they are called multiple edges. A graph $G$ is said to be simple if $G$ has neither loops nor multiple edges. In this thesis, we only deal with simple graphs unless otherwise stated. The degree of a vertex $x$ is the number of edges incident with $x$, which is denoted by $\deg(x)$, and a $k$-vertex is a vertex of degree $k$. Moreover, for a graph $G$, the maximum degree and the minimum degree of $G$, denoted by $\Delta(G)$ and $\delta(G)$, respectively, are defined as follows:

$$\Delta(G) = \max\{\deg(x) \mid x \in V(G)\}$$

$$\delta(G) = \min\{\deg(x) \mid x \in V(G)\}$$
The set of vertices of $G$ adjacent to a vertex $x \in V(G)$ is called the \textit{neighborhood} of $x$ in $G$ and is denoted by $N(x)$. It is clear that if a graph $G$ is simple, then $\deg(x) = |N(x)|$ for each $x \in V(G)$. A graph $G$ is called \textit{$k$-regular} if for each $v \in V(G)$, $\deg(v) = k$. For any graph, the following holds.

**Theorem 1.1 (Handshaking Lemma)** In any graph $G$, the following equation holds:

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

Moreover, by this theorem, we immediately have the following corollary.

**Corollary 1.2 (Odd Point Theorem)** Every graph has an even number of vertices of odd degree.

Two simple graphs $G$ and $H$ are said to be \textit{isomorphic}, denoted by $G \cong H$, if there is a bijection $\sigma : V(G) \rightarrow V(H)$ such that $xy \in E(G)$ if and only if $\sigma(x)\sigma(y) \in E(H)$ for any $x, y \in V(G)$. In this case, a bijection $\sigma$ is called an \textit{isomorphism} between $G$ and $H$.

Let $G$ be a graph and let

$$W := x_1e_1x_2e_2 \cdots e_kx_{k+1},$$

where for $x_i \in V(G)$ and $e_i \in E(G)$, each $e_i$ joins $x_i$ and $x_{i+1}$ for $i = 1, 2, \ldots, k$. Then the sequence $W$ is called a \textit{walk} in $G$, and $x_1$ and $x_{k+1}$ are called the \textit{ends} of $W$. The number of $k$ is called the \textit{length} of $W$ and is denoted by $|W|$. If $x_1, \ldots, x_{k+1}$ are all distinct, then $W$ is called a \textit{path} in $G$. Moreover, we sometimes call a path $P$ (or a walk $W$) a $(u, v)$-\textit{path} (or a $(u, v)$-\textit{walk}) if $x_1 = u$ and $x_{k+1} = v$. In the walk $W$, if $x_1 = x_{k+1}$, then $W$ is called \textit{closed}. A closed walk $W$ is called a \textit{cycle} if $x_1, \ldots, x_{k+1}$ are all distinct. We call a cycle of length $k$ a $k$-\textit{cycle}. In this case, if $k$ is even (resp., odd), then we sometimes call a $k$-cycle an even (resp., odd) \textit{cycle}. A cycle $C$ is said to be a \textit{Hamilton cycle} if $C$ passes through each vertex of $G$ exactly once. A graph $G$ is said to be \textit{Hamiltonian} if $G$ has a Hamilton cycle. A \textit{wheel} is a simple graph obtained from a cycle $C$ of length at least 3 and a single vertex $v$ by joining $v$ and all vertices of $C$.

For two graphs $G$ and $H$, $H$ is said to be a \textit{subgraph} of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K$ be a subgraph of a graph $G$ and let $S$ be a nonempty subset of $V(G)$. If $V(K) = S$ and $E(K)$ consists of the edges in $E(G)$ whose both ends are in $S$, then the subgraph $K$ is said to be \textit{induced} by $S$, denoted by $(S)$.

Let $G$ be a graph and let $x \in V(G)$ and $e = uv \in E(G)$. For a subset $W \subseteq V(G)$, we define $G - W = (V(G) - V(W))$. In particular, if $W = \{x\}$, then we denote simply by $G - x$. Similarly, for a subgraph $H$ of $G$, we define $G - H = (V(G) - V(H))$. The graph $G/e$ (resp., $G - e$) is obtained from $G$ by \textit{contracting} (resp., \textit{removing}) the edge $e$. To get $G/e$, we identify the vertices $u$ and $v$ and remove all resulting loops and multiple edges. For examples of the above definitions, see Figure 1.2.

A graph is \textit{connected} if any two of its vertices can be joined by a path, and otherwise it is \textit{disconnected}. Moreover, a graph $G$ is \textit{$k$-connected} if and only if the graph obtained
from $G$ by removing $k - 1$ vertices in $V(G)$ is connected. A cut set is a vertex set $S$ such that $G - S$ is disconnected. In particular, for a connected graph $G$, if there exists $x \in V(G)$ such that $G - x$ is disconnected, then $x$ is said to be a cut vertex.

A graph $G$ is a tree if and only if $G$ has no cycle. In this case, if $G$ is disconnected, then $G$ is called a forest. For every tree, the following holds.

**Theorem 1.3** If a graph $G$ is a tree, then $|E(G)| = |V(G)| - 1$.

*Proof.* We prove the theorem by the induction on $|V(G)|$. If $|V(G)| = 1$, then $G$ is a tree with exactly one vertex and no edge. Suppose that the theorem holds for any tree $G$ with $|V(G)| < k$, and we consider the case when $|V(G)| = k$. Let $uv \in E(G)$. Now, $G - uv$ has no $(u, v)$-path (otherwise, $G$ clearly has a cycle), that is, $G - uv$ is disconnected. Since two components $G_1, G_2$ in $G - uv$ have no cycle, they are trees. Hence, since $|G_1| < k$ and $|G_2| < k$, by the induction hypothesis, we have

$$|E(G_1)| = |V(G_1)| - 1, \quad |E(G_2)| = |V(G_2)| - 1$$

$$|E(G)| = |E(G_1)| + |E(G_2)| + 1 = |V(G_1)| + |V(G_2)| - 1 = |V(G)| - 1.$$

For two graphs $G$ and $H$, where $H$ is a subgraph of $G$, if $V(H) = V(G)$, then $H$ is said to be a spanning subgraph of $G$. In particular, if a spanning subgraph of $G$ is a tree, then the subgraph is called a spanning tree.

**Theorem 1.4** Every connected graph $G$ has a spanning tree.
Proof. If \( G \) is a tree, then we are done since \( G \) is also a spanning tree of \( G \). Hence we suppose that \( G \) is not a tree, that is, \( G \) has a cycle \( C \). Let \( e \in E(G) \) be an edge contained in \( C \), and let \( H = G - e \). If \( H \) is a tree, then we are done. Otherwise, \( H \) has a cycle \( C' \), and then, we consider \( H' = H - e' \) similarly to the above, where \( e' \in E(H) \) is an edge contained in \( C' \). By repeating this argument, we finally obtain a spanning tree of \( G \). \( \blacksquare \)

A \( k \)-coloring of \( G \) is a map \( c : V(G) \to \{1, 2, \ldots, k\} \) such that for any edge \( uv \in E(G) \), \( c(u) \neq c(v) \). A graph \( G \) is \( k \)-colorable if there exists a \( k \)-coloring of \( G \). The chromatic number \( \chi(G) \) is the minimum number \( k \) such that \( G \) is \( k \)-colorable. In particular, we sometimes call a graph \( G \) with \( \chi(G) = k \) a \( k \)-chromatic graph.

Clearly, for every graph \( G \) with \( n \) vertices, \( \chi(G) \leq n \). Moreover, it is not difficult to see that for every connected graph \( G \), \( \chi(G) \leq \Delta(G) + 1 \). (We can prove the statement by the induction on the number of vertices.) However, Brooks [13] proved that graphs \( G \) with \( \chi(G) = \Delta(G) + 1 \) do not exist so much as follows. A simple graph \( G \) is said to be complete if every pair of vertices of \( G \) are adjacent. The complete graph with \( n \) vertices is denoted by \( K_n \). Note that \( \chi(K_n) = n = \Delta(K_n) + 1 \).

**Theorem 1.5 (Brooks [13])** For every connected graph \( G \) that is not an odd cycle or a complete graph,

\[
\chi(G) \leq \Delta(G). 
\]

In a graph \( G \), the distance of two vertices \( x, y \in V(G) \), denoted by \( d_G(x, y) \), the length of a shortest path which connects \( x \) and \( y \) in \( G \). (If \( G \) has no path which connects \( x \) and \( y \), then we define \( d_G(x, y) := \infty \).) Moreover, the diameter of \( G \), denoted by \( \text{diam}(G) \), is defined as follows;

\[
\text{diam}(G) := \max \{d_G(x, y) \mid x, y \in V(G)\}.
\]

A graph \( G \) is bipartite if there exists a partition \( V(G) = V_1 \cup V_2 \) and \( V_1 \cap V_2 = \emptyset \) such that any two vertices in the same set \( V_i \) are not adjacent for \( i = 1, 2 \). Moreover, it is known that the partition of any bipartite graph is uniquely determined. A graph is said to be the complete bipartite graph, denoted by \( K_{m,n} \), if it is a bipartite graph such that \( |V_1| = m \) and \( |V_2| = n \) and every pair of a vertex in \( V_1 \) and a vertex in \( V_2 \) is joined. In particular, if \( m = 1 \) or \( n = 1 \), then \( K_{m,n} \) is called a star. By the above definition, since every bipartite graph \( G \) is 2-colorable, \( V(G) \) can be colored by two colors, black and white. (It is clear that if \( G \) is 2-colorable, then \( G \) is bipartite.) Therefore, in a bipartite graph \( G \), we denote the set of black vertices and that of white ones by \( V_B(G) \) and \( V_W(G) \), respectively. Moreover, let the bipartition size mean \( (|V_B(G)|, |V_W(G)|) \) in a bipartite graph \( G \). Clearly, every bipartite graph has no odd cycle. In fact, bipartite graphs are characterized by this property.

**Theorem 1.6** A graph \( G \) is bipartite if and only if \( G \) has no odd cycle.

**Proof.** It suffices to prove that if a graph \( G \) has no odd cycle, then \( G \) is bipartite. Let \( G \) be a graph containing no odd cycle, and we may suppose that \( G \) is connected. We fix a vertex \( x \in V(G) \). For any vertex \( v \in V(G) \), we color \( v \) by black if \( d_G(x, v) \) is even, and color \( v \) by white if \( d_G(x, v) \) is odd. (Note that \( x \) is colored by black since \( d_G(x, x) = 0 \).)
Since every distance is even or odd, all vertices in $G$ are colored by black or white. Let $e = uv \in E(G)$. Let $P = x_{y_1y_2}\ldots u$ be a shortest path connecting $x$ and $u$, and let $P' = x_{z_1z_2}\ldots v$ be that connecting $x$ and $v$. It suffices to prove that $|P| \not\equiv |P'| \pmod 2$. Then, let $s$ be the last vertex among vertices shared by both $P$ and $P'$ (see Figure 1.3).

![Figure 1.3: The last vertex $s$](image)

If $s$ is $u$ or $v$, then we are done since $d_G(s, u) = d_G(s, v) - 1$ or $d_G(s, u) = d_G(s, v) + 1$. Otherwise, that is, if $s \neq u, v$, then the length of the cycle in Figure 1.3 is equal to $d_G(s, u) + 1 + d_G(s, v)$. By the assumption, the length is even. Then, $d_G(s, u)$ and $d_G(s, v)$ have different parity. On the other hand, we now have

$$d_G(x, u) = d_G(x, s) + d_G(s, u) \text{ and } d_G(x, v) = d_G(x, s) + d_G(s, v).$$

Therefore, $x$ and $y$ receive different colors, that is, $G$ is bipartite. ■

### 1.2 Planar graphs

In this section, we shall give several preliminaries for planar graphs. A graph $G$ is planar if $G$ can be drawn on the plane (resp., the sphere) without crossing edges, where this drawing is called an embedding of $G$ on the plane (resp., the sphere). For example, the graph in Figure 1.1 is planar (we can re-draw the graph on the plane without crossing edges). We sometimes call a graph embedded on the plane (resp., the sphere) a plane graph (resp., a sphere graph). Every plane (resp., sphere) graph cuts and divides the plane (resp., the sphere) into several regions. Each region of a plane (or a sphere) graph $G$ is called a face, and the set of faces is denoted by $F(G)$. Note that any plane graph has a unique outer region which has no boundary.

Firstly, we introduce Jordan Curve Theorem and Schönflies Theorem. They are the most basic fact for considering the topology of the plane. In particular, we often use Jordan Curve Theorem in the proof of our main results. A simple closed curve means the 1-dimensional sphere on the plane, that is, $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ (this word will be exactly defined in the next section).

**Theorem 1.7 (Jordan Curve Theorem [77])** Any simple closed curve $C$ on the plane divides the plane into exactly two connected components, the interior and the exterior. Both of these regions have $C$ as the boundary.
Theorem 1.8 (Schönflies Theorem [75]) The interior of any simple closed curve on the plane is homeomorphic to an open 2-cell.

By using the stereographic projection (which is a particular map projecting the sphere onto the plane), we can regard any face of $G$ embedded on the sphere as an outer region of $G$ embedded on the plane, that is, we can deal with embeddings of graphs on the sphere as those on the plane. Thus, following this, we often regard the two embeddings as the same.

Figure 1.4 shows that embeddings of Platonic solids (or convex regular polyhedrons). By these figures, we can obtain Table 1.1, where $|V|$, $|E|$ and $|F|$ denote the number of vertices, edges and faces of Platonic solids, respectively.

![Embeddings of Platonic solids on the plane](image)

| Solid      | $|V|$ | $|E|$ | $|F|$ |
|------------|------|------|------|
| Tetrahedron| 4    | 6    | 4    |
| Cube       | 8    | 12   | 6    |
| Octahedron | 6    | 12   | 8    |
| Dodecahedron| 20   | 30   | 12   |
| Icosahedron| 12   | 30   | 20   |

Table 1.1: The number of vertices, edge and faces of Platonic solids

In this table, we can easily check that $|V| - |E| + |F| = 2$ holds for every Platonic solid. In 1752, Euler found that the equation always holds for every plane graph.
**Theorem 1.9 (Euler’s formula)**  For any connected plane graph $G$, the following holds:

$$|V(G)| - |E(G)| + |F(G)| = 2$$

**Proof.** Let $G$ be a plane graph with $p$ vertices, $q$ edges and $r$ faces. Since $G$ is connected, $G$ has a spanning tree $T$ by Theorem 1.4. Note that since $|V(T)| = p$, $|E(T)| = p - 1$ and $|F(T)| = 1$ and $p - (p - 1) + 1 = 2$, the equation holds for $T$. Suppose that the equation holds for a subgraph $H$ of $G$ containing $T$, and let $e \in E(G)$ be an edge which is not in $E(H)$.

In $H + e$, if $e$ is shared by exactly one face, then it is not difficult to see that $H$ is disconnected since there exists a simple closed curve crossing only $e$ with $H + e$, which separates $H$ into two connected components. However, since $H$ contains a spanning tree $T$, this is a contradiction. Hence, $e$ is shared by two distinct faces $f$ and $f'$ in $H + e$.

On the other hand, if $f$ and $f'$ are merged into one face $f''$ in the embedding of $H$. Hence, by adding $e$ into $f''$, both the number of edges and that of faces increase by exactly one. By the above operation, the value of the equation “$=2$” is unchanged. Therefore, by repeating this operation, we have $p - q + r = 2$. ■

Euler’s formula gives us one guidance to decide whether a given graph is planar. For example, we can easily prove the following lemma.

**Lemma 1.10** $K_5$ and $K_{3,3}$ are not planar.

**Proof.** Suppose that $K_5$ and $K_{3,3}$ are planar, and let $F_1$ and $F_2$ be the number of faces in embeddings of $K_5$ and $K_{3,3}$ on the plane, respectively. By Euler’s formula, we have

$$5 - 10 + F_1 = 2, \quad 6 - 9 + F_2 = 2. \quad \cdot \cdot \cdot (1)$$

Moreover, the size of each face is at least 3 for $K_5$ and that is at least 4 for $K_{3,3}$ (note that $K_{3,3}$ is bipartite). Hence, we have

$$2 \times 10 \geq 3F_1, \quad 2 \times 9 \geq 4F_2.$$

However, these inequalities contradict (1). ■

In particular, there is a very famous characterization of planar graphs which is well known as “Kuratowski’s Theorem” as follows. (Since the proof of Theorem 1.11 is a little difficult, we omit it in this thesis.) A subdivision of a graph $G$ is obtained from $G$ by replacing edges of $G$ with paths of length at least 1. Note that $G$ is also a subdivision of $G$.

**Theorem 1.11 (Kuratowski’s Theorem)** A graph $G$ is planar if and only if $G$ contains no subdivisions of $K_5$ and $K_{3,3}$ as its subgraphs.

Moreover, we introduce following theorems, each of which has an important relation with our main topics.
Theorem 1.12 A connected plane graph $G$ is 2-connected if and only if each face of $G$ is bounded by a cycle.

Proof. For a connected plane graph $G$, it is not difficult to see that $x \in V(G)$ is a cut vertex if and only if there exists a simple closed curve $C$ on the plane such that it intersects only with $x$ and divides $G$ into its interior and its exterior. In this case, for a face $F$ including $C$ in the interior, $x$ appears on the boundary walk of $F$ at least two times. Conversely, if a 2-connected graph $G$ has a face $F$ such that the boundary of $F$ is not a cycle, then $G$ has a simple closed curve $C$ in the interior of $F$ such that it intersects only $v$, where $v$ is a vertex appearing on the boundary of $F$ at least two times. In this case, by Jordan Curve Theorem, $C$ separates $G - v$ into two connected components, which contradicts the 2-connectivity of $G$. Therefore, this theorem holds.

Every vertex $v$ in a plane graph $G$ is included in the interior of a face $f \in F(G - v)$. The boundary walk of $f$ is called a link of $v$. Simply put, the link of $v$ is the boundary of the union of faces which are incident with $v$.

Theorem 1.13 A connected plane graph $G$ is 3-connected if and only if the link of $v$ is a cycle for any vertex $v$ in $G$.

Proof. A plane graph $G$ is 3-connected if and only if $G - v$ is 2-connected for any vertex $v$. Therefore, by Theorem 1.12, it is equivalent to that the link of $v$ is a cycle.

For a given plane graph $G$, we construct a dual graph of $G$ as follows: A vertex is placed in each face of $G$, and two distinct vertices are joined by an edge for each common edge on the boundaries of the two corresponding faces of $G$. Lastly, by deleting $G$, we obtain a dual graph of $G$ (see Figure 1.5). An even-embedding $G$ on the plane is a plane graph such that each face of $G$ is bounded by an even cycle (this word will be exactly defined in the next section).

![Figure 1.5](image-url)

Figure 1.5: $G^*$ is a dual graph of $G$, and $G^*$ is planar since each edge of $G^*$ is drawn so that it crosses only its associated edge of $G$.

Theorem 1.14 Every even-embedding on the plane is bipartite.
Proof. Let $G$ be an even-embedding on the plane. By Theorem 1.6, if $G$ is bipartite, then $G$ has no odd cycle $C$. Thus, we may suppose that $G$ has an odd cycle $C$. Since $G$ is an even-embedding, $C$ does not bound a face. Then we consider a plane graph $G'$ consisting of $C$ and vertices and edges in the interior of $C$. An infinite face of $G'$ is bounded by an odd cycle and each finite face of $G'$ is bounded by an even cycle. Thus, for a dual graph of $G'$, a vertex corresponding to an outer region has odd degree and each vertex corresponding to any finite face has even degree, which contradicts to Corollary 1.2.

Finally, we introduce a very famous theorem for planar graphs which is called the *four color theorem*. This theorem was first proved by Kempe in 1879, however, this proof was shown incorrect by Heawood in 1890. After that, the theorem was unsolved for a long time, which is well known as the *four color problem*. Then, in 1976, the problem was finally solved by Appel and Haken by using a computer.

**Theorem 1.15 (Four color theorem [3, 4, 5])** Every planar graph is 4-colorable.

However, in Appel and Haken’s proof, a computer is used for a long time, and the logical proof has not yet been known. In Chapter 5, we introduce one possibility to prove the four color theorem by a logical proof using a local transformation.

### 1.3 Embeddings

Throughout this thesis, we shall call a connected compact 2-dimensional manifold without boundaries a *closed surface*. There are two classes of closed surfaces, *orientable* ones and *non-orientable* ones. On an orientable closed surface, we can compatibly prescribe clockwise and counter clockwise orientations around all the points on it. On the other hand, we cannot do on non-orientable closed surfaces. For example, on a Möbius band, we cannot give compatible clockwise orientations to points on the center line of the Möbius band (see Figure 1.6). In fact, a closed surface is orientable if and only if it does not include a Möbius band.

![Figure 1.6: Möbius band](image)

Let $F^2_1$ and $F^2_2$ be two closed surfaces. The closed surface obtained from $F^2_1$ with a disk removed and $F^2_2$ with a disk removed by pasting them along their boundaries is called a *connected sum* of $F^2_1$ and $F^2_2$, denoted by $F^2_1 \# F^2_2$. We can characterize orientable and non-orientable closed surfaces, as follows. A closed surface is an *orientable* surface of genus...
$g$, denoted by $S_g$, if $F^2$ is homeomorphic to $T^2 \# \cdots \# T^2$, where $T^2$ is the torus. On the other hand, a closed surface is a non-orientable surface of genus (or cross-cap number) $k$, denoted by $N_k$, if $F^2$ is homeomorphic to $P^2 \# \cdots \# P^2$, where $P^2$ is the projective plane. Equivalently, $N_k$ is obtained from the sphere with $k$ pairwise disjoint disk removed by attaching $k$ Möbius bands to each boundary of the punctured sphere. For example, $S_0, S_1, N_1$ and $N_2$ are the sphere, the torus, the projective plane and the Klein bottle, respectively.

By the classification theorem of closed surfaces, it is known that every closed surface is homeomorphic to either an orientable surface or a non-orientable surface with some genus. For non-orientable closed surfaces, it is also known that $N_3$ and $N_4$ are homeomorphic to $T^2 \# P^2$ and $T^2 \# K^2$, respectively, where $K^2$ stands for the Klein bottle. In general, for any positive integer $k$ and any even integer $0 \leq k' < k$, $N_k$ is homeomorphic to $N_{k-k'} \# S^{g'}_2$.

A closed curve on a closed surface $F^2$ is a continuous function $l : S^1 \rightarrow F^2$ or its image, where $S^1$ is the 1-dimensional sphere, that is, $\{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$. A closed curve $l$ is called simple if the function $l$ is an injection. A simple closed curve $l$ on a closed surface $F^2$ is called separating (resp., non-separating) if $F^2 - l$ is disconnected (resp., connected). A simple closed curve $l$ on a closed surface $F^2$ is said to be trivial (or contractible) if $l$ bounds a 2-cell on $F^2$. Otherwise, $l$ is said to be essential (or non-contractible). Among essential simple closed curves, one with an annular neighborhood is called 2-sided while one whose tubular neighborhood forms a Möbius band is called 1-sided. Two closed curves $l_1$ and $l_2$ on a closed surface $F^2$ are said to be homotopic to each other on $F^2$ if there exists a continuous function $\phi : [0, 1] \times S^1 \rightarrow F^2$ such that $\phi(0, x) = l_1(x)$ and $\phi(1, x) = l_2(x)$ for each $x \in S^1$.

When we discuss embeddings of graphs into surfaces, we regard graphs as 1-dimensional topological spaces, not only as combinatorial objects. Roughly speaking, to embed a graph into a surface $F^2$ is to draw the graph on $F^2$ without crossing edges. It is sometimes effective to regard an embedding as an injective continuous map $f : G \rightarrow F^2$. We deal with $G$ and $f(G)$ as the same object intuitively. However, to distinguish $G$ from the embedded one $f(G)$, we sometimes call $G$ an abstract graph while we call $f(G)$ an embedding. In this thesis, we often denote an embedded graph by $G$.

When $G$ is embedded in a closed surface $F^2$, $G$ can be regarded as a subset of $F^2$. Each component of $F^2 - G$ is called a face of $G$ embedded in $F^2$. A closed walk $W$ (resp., cycle $C$) of $G$ which bounds a face $F$ of $G$ is called the boundary walk (resp., boundary cycle) of $F$. An embedded graph $G$ is said to be a 2-cell embedding, or $G$ is said to be 2-cell embedded in $F^2$ if each face of $G$ is homeomorphic to an open 2-cell, that is, $\{(x, y) \in R^2 \mid x^2 + y^2 < 1\}$. After this, we simply call 2-cell embeddings embeddings. An even-(resp., odd-)embedding on a closed surface is a graph such that each face is bounded by a cycle of length even (resp., odd). For a graph $G$ embedded on a closed surface $F^2$, we denote the face set of $G$ by $F(G)$, and denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. Moreover, for any face (or a 2-cell region) $f$ in a graph $G$ on a closed surface, $\partial f$ denotes the boundary walk of $f$.

Let $G_1$ and $G_2$ be two graphs embedded on closed surfaces $F^2_1$ and $F^2_2$, respectively. Two graphs $G_1$ and $G_2$ are said to be homeomorphic to each other if there exists a
homeomorphism $h: F_1^2 \to F_2^2$ with $h(G_1) = G_2$ which induces an isomorphism from $G_1$ to $G_2$. In this case, we also say that $G_1 \subset F_1^2$ and $G_2 \subset F_2^2$ are the same ones up to homeomorphism.

So far, we have not referred to the orientability of surfaces or used Euler’s formula. To make it explicit, the Euler characteristic $\varepsilon(F^2)$ of a closed surface $F^2$ is defined as

$$\varepsilon(F^2) = \begin{cases} 2 - 2g & \text{(if $F^2 = S_g$)}, \\ 2 - k & \text{(if $F^2 = N_k$)}. \end{cases}$$

Note that if $F^2$ is the sphere (or the plane), then $\varepsilon(F^2) = 2$, where the equation is introduced in the previous section. Finally, we introduce the following theorem is well known as “Euler’s formula”. (Throughout this thesis, Euler’s formula means the following equation.)

**Theorem 1.16 (Euler’s formula)** Let $G$ be a graph (might not be simple) which is embedded in a closed surface $F^2$. Then, the following holds:

$$|V(G)| - |E(G)| + |F(G)| = \varepsilon(F^2)$$
Chapter 2

Equivalence for $N$-angulations by diagonal transformations

In this chapter, we describe equivalence for $N$-angulations on closed surfaces by diagonal transformations.

2.1 Definitions

An $N$-angulation $G$ on a closed surface $F^2$ is an embedding of a 2-connected simple graph on $F^2$ such that every face of $G$ is bounded by a cycle of length $N$, where $N \geq 3$ is an integer. In particular, if $N = 3, 4, 5$ or $6$, then the corresponding graph is called a triangulation, a quadrangulation, a pentangulation or a hexangulation, respectively. Note that if $N$ is even, then every $N$-angulation on the sphere is always bipartite by Theorem 1.14. Moreover, an $N$-angulation on a closed surface $F^2$ is simply said to be an $N$-angulation if we do not need to specify $F^2$. In an $N$-angulation, let $P = x_1y_1y_2\cdots y_{l-2}y_{l-1}x_k$ be a path of length $l$ ($1 \leq l \leq \lfloor \frac{N}{2} \rfloor$, $l + k = N + 1$), which is shared by two faces $F_1$ and $F_2$, where $\partial F_1 = x_1x_2\cdots x_{k-1}x_ky_{l-1}\cdots y_2y_1$ and $\partial F_2 = x_1x_2x_3\cdots x_{k+1}x_ky_{l-1}\cdots y_2y_1$, respectively. Replacing a path $P$ with a path $P' = x_1y_1y_2\cdots y_{l-2}y_{l-1}x_{k+1}$ is called a diagonal transformation (see Figure 2.1). Note that there are two choices of diagonal transformations, that is, replacing $P$ with a path $P'' = x_2x_3y_2\cdots y_{l-2}y_{l-1}x_k$ is also a diagonal transformation in an $N$-angulation.

![Figure 2.1: A diagonal transformation in an $N$-angulation](image)

So, for $N$-angulations, there are $\lfloor \frac{N}{2} \rfloor$ kinds of diagonal transformations, depending
on the length \( l \) of the path \( P \). When this transformation breaks the simpleness or 2-connectedness of graphs, we don’t apply it.

Two \( N \)-angulations are said to be equivalent if they can be transformed into each other by diagonal transformations, up to homeomorphism. (Two graphs \( G \) and \( H \) can be transformed into each other by several transformations means that we can transform \( G \) into \( H \) by applying the transformations to \( G \) repeatedly.)

### 2.2 Triangulations

In this section, we shall introduce the results on diagonal transformations in triangulations, done by Wagner, Dewdney, Negami and Watanabe. Moreover, we also describe the outline of the proof of Negami’s general result.

#### 2.2.1 Classical results

By the definition of diagonal transformations, there is only one kind of diagonal transformations in triangulations, specially called a diagonal flip (see Figure 2.2).

![Figure 2.2: A diagonal flip](image)

The following theorem, which we shall refer as Wagner’s theorem, is the starting point of our studies on diagonal transformations in \( N \)-angulations. However, since Wagner’s paper [78] is written in German, we introduce the proof here. (This proof can be also found in [62].) The standard form \( \Delta_n \) of triangulations on the sphere is a triangulation shown in Figure 2.3 with \( n + 3 \) vertices.

**Theorem 2.1 (Wagner [78])** Any two triangulations on the sphere with the same number of vertices are equivalent.

**Proof.** In the proof, we transform any triangulation on the sphere with \( n + 3 \) vertices into the standard form \( \Delta_n \) by diagonal flips. If the above is true, then any two triangulations on the sphere with \( n + 3 \) vertices can be transformed into each other by diagonal flips through the standard form \( \Delta_n \).

Let \( G \) be a triangulation on the sphere with \( n + 3 \) vertices and fix the outer region \( xyz \) of \( G \). Note that if \( n = 0 \) (resp., \( n = 1 \)), then we are done since \( G \cong K_3 \) (resp., \( G \cong K_4 \)), and hence, we may suppose that \( n \geq 2 \). First, suppose that \( \deg(x) \geq 4 \), and let \( y, v_1, v_2, \ldots, v_k = z \) be the neighbors of \( x \) lying around \( x \) on this cyclic order. If \( y \) is not adjacent to \( v_2 \) in \( G \), then we can replace \( xv_1 \) with \( yv_2 \) by applying a diagonal flip to
Otherwise, since we now have an edge $yv_2$ in $G$, $v_1$ and $v_3$ (which is possibly $z$) are not adjacent by Jordan Curve Theorem. Thus, we can apply a diagonal flip to $xv_2$ to decrease $\deg(x)$. In either case, we can decrease $\deg(x)$ by a diagonal flip, and hence, we can finally obtain $\deg(x) = 3$.

Now, we may suppose that $\deg(x) = 3$ and let $w$ is the unique neighbor of $x$ except $y$ and $z$. Then, we next consider the subgraph $G'$ with the outer cycle $wyz$. In this case, if $\deg(w) = 3$, then we are done since $G \cong \Delta_1$. Carrying out the same argument on $G'$ as the above, we apply diagonal flips to $G'$ until we obtain $\deg(w) = 4$. By repeating the above operations, we can transform $G$ into the standard form $\Delta_n$. ■

Figure 2.3: The standard form $\Delta_n$ of triangulations on the sphere

The proof of Theorem 2.1 strongly depends on the planarity of triangulations. The point is how to make a vertex of degree 3. After making a vertex of degree 3, we can consider another triangulation with fewer vertices than the original one, that is, we can use the inductive argument on the number of vertices. The following two lemmas enable us to do it, where these lemmas are clear by Figures 2.4 and 2.5, respectively. (For the details, see [61] or [68].)

**Lemma 2.2** Let $G$ be a triangulation. A vertex of degree 3 in $G$ can be moved to any face of $G$ by diagonal flips.

**Lemma 2.3** Let $G$ and $G'$ be triangulations with vertices $v$ and $v'$ of degree 3, respectively. A sequence of diagonal flips from $G - v$ to $G' - v'$ can be translated into that from $G$ to $G'$ using the vertices $v$ and $v'$.

Figure 2.4: Moving a vertex of degree 3 (Lemma 2.2)
By the above lemmas, if we can make a vertex of degree 3 in two given triangulations, then we can prove that they are equivalent by the induction hypothesis. Thus, it suffices that we consider triangulations in which we cannot make a vertex of degree 3 to establish the first step of the induction.

Let $G$ be a triangulation on a closed surface $F^2$. Then $G$ is said to be \textit{minimal} if $G$ is the smallest among all triangulations on $F^2$, and $G$ is said to be \textit{pseudo-minimal} if $G$ cannot be transformed into a triangulation which has a vertex of degree 3. It is easy to see that if a triangulation $G$ is minimal, then $G$ is pseudo-minimal. When $G$ is a complete graph, $G$ is clearly pseudo-minimal since any two vertices of $G$ are adjacent. Moreover, it is known that if a complete graph triangulates a closed surface, then the triangulation must be minimal [71, 72].

Let $G$ be a triangulation on a closed surface $F^2$ and let $\Delta_n$ be the standard form of triangulations on the sphere (see Figure 2.3). We denote the triangulation obtained from $G$ by adding $\Delta_n$ to a face of $G$ by $G + \Delta_n$. It is clear that $|V(G + \Delta_n)| = |V(G)| + n$. By Lemma 2.2, $G + \Delta_n$ denotes the unique triangulation up to equivalence. Moreover, it is easy to see that $G + \Delta_m + \Delta_n$ is equivalent to $G + \Delta_{m+n}$ by Lemma 2.2.

By Lemmas 2.2 and 2.3, we can prove the similar statement to Wagner’s result using the following inductive argument: Let $G_1$ and $G_2$ be triangulations on a closed surface $F^2$ with the same number of vertices. Suppose that for $i = 1, 2$, $G_i$ can be transformed into a triangulation $G'_i$ which has a vertex of degree 3. Then, since $G'_1$ and $G'_2$ are equivalent by the hypothesis of induction, thus $G_1$ and $G_2$ are also by Lemma 2.3. In this case, if a closed surface $F^2$ admits only one pseudo-minimal triangulation up to homeomorphism, say $T$, then we can immediately conclude that any two triangulations $G_1$ and $G_2$ on $F^2$ with the same number of vertices can be transformed into each other by diagonal flips through $T + \Delta_n$, where $n = |V(G_i)| - |V(T)|$. If all pseudo-minimal triangulations are equivalent, then we can also show the same fact.

It is known that the torus admits only one minimal triangulation, which is a complete
graph $K_7$ shown in Figure 2.6. (In the figure, the rectangle represents the torus by identifying opposite sides in parallel respectively.) Hence, the following theorem has been proved.

![Figure 2.6: The minimal triangulation $K_7$ on the torus](image)

**Theorem 2.4 (Dewdney [20])** Any two triangulations on the torus with the same number of vertices are equivalent.

It was proved that there exists a unique minimal triangulation on the projective plane, which is a complete graph $K_6$, and there exist exactly six minimal triangulations on the Klein bottle. Thus, similarly to Theorem 2.4, the following theorems are proved.

**Theorem 2.5 (Negami and Watanabe [68])** Any two triangulations on the projective plane with the same number of vertices are equivalent.

**Theorem 2.6 (Negami and Watanabe [68])** Any two triangulations on the Klein bottle with the same number of vertices are equivalent.

Note that these arguments strongly depend on what the surface is, and hence, it seems to be difficult to extend the theorem to general closed surfaces.

### 2.2.2 General case

As is mentioned in the previous subsection, for a given closed surface $F^2$, since determining the pseudo-minimal triangulations on $F^2$ is strongly depending on the topology of $F^2$, it seems to be difficult to make Wagner’s theorem for general closed surfaces. However, Negami found a nice breakthrough for the extension and proved the following theorem. In the remainder of this subsection, we introduce Negami’s idea and the outline of the proof of Theorem 2.7.

**Theorem 2.7 (Negami [61])** For any closed surface $F^2$, there exists a positive integer $N(F^2)$ such that any two triangulations $G$ and $G'$ on $F^2$ are equivalent if $|V(G)| = |V(G')| \geq N(F^2)$. 

29
First, let us consider the generating of triangulations. A contraction of an edge $e$ in a triangulation $G$ is to shrink $e$ on the surface and to eliminate diagonal regions as shown in Figure 2.7. The resulting graph is denoted by $G/e$. An edge $e$ is said to be contractible if $G/e$ is also a triangulation. Moreover, an edge $e$ is not contractible if and only if $e$ lies on a 3-cycle which does not bound a face, and we do not contract an edge $e$ if it is not contractible. A triangulation $G$ is said to be contractible to another triangulation $T$ if $T$ can be obtained from $G$ by a sequence of contractions of edges. In this case, note that $|V(T)| = |V(G)| - m$ if we apply contractions of edges $m$ times. Moreover, a triangulation $G$ is said to be irreducible if no edge of $G$ is contractible.

There are many results on irreducible triangulations. In particular, for closed surface with non-negative Euler characteristics, all the irreducible triangulations have already been determined as follows. (In Theorem 2.8, the authors deal with $K_4$ as the minimal triangulation of the sphere.)

**Theorem 2.8 (Rademacher and Steinitz [70])** There exists only one irreducible triangulation on the sphere, which is $K_4$.

**Theorem 2.9 (Barnette [8])** There exist precisely two irreducible triangulations on the projective plane.

**Theorem 2.10 (Lawrencenko [40])** There exist precisely 21 irreducible triangulations on the torus.

**Theorem 2.11 (Lawrencenko and Negami [41], Sulanke [74])** There exist precisely 29 irreducible triangulations on the Klein bottle.

Moreover, for general closed surfaces, Nakamoto and Ota [56] found the upper bound of the number of vertices of irreducible triangulations as follows; which implies that the number of irreducible triangulations is finite. (The number in the following theorem is recently improved by Joret and Wood [35].)

**Theorem 2.12 (Nakamoto and Ota [56])** For any closed surface $F^2 \neq S_0$ and any irreducible triangulation $G$ on $F^2$, the following holds:

$$|V(G)| \leq 270 - 171\varepsilon(F^2)$$
Next, let us consider the relation between diagonal flips and contractions of edges. Any diagonal flip does not change the number of vertices of a triangulation $G$, but any contraction of an edge decreases it by exactly one. Although this impresses us with their difference, the following lemma connects them. It is easy to see that the following holds by Figure 2.8.

**Lemma 2.13** Contraction of an edge $e$ in a triangulation $G$ can be realized as a sequence of diagonal flips followed by removing a vertex of degree 3. □

![Figure 2.8: A contraction of an edge $e$ and diagonal flips](image)

Then, the following lemma immediately follows by Lemmas 2.2, 2.3 and 2.13, which is a key lemma to prove Theorem 2.7.

**Lemma 2.14** Let $G$ and $T$ be triangulations on a closed surface $F^2$. Then, if $G$ can be contractible to $T$, then $G$ is equivalent to $T + \Delta_m$, where $m = |V(G)| - |V(T)|$.

Finally, we need a more key fact to prove Theorem 2.7: Let $G_1$ and $G_2$ be two triangulations on a closed surface $F^2$. We claim that there exists a pair of integers $m_1$ and $m_2$ such that $G_1 + \Delta_{m_1}$ and $G_2 + \Delta_{m_2}$ are equivalent. Here, $G_1$ and $G_2$ are not supposed to have the same number of vertices. Embed $G_1$ and $G_2$ on $F^2$ simultaneously so that they intersect at only their edges, and put a vertex on each intersection of them. Add several edges to the resulting graph to be a triangulation on $F^2$, say $G$. Showing that $G$ is contractible to both $G_1$ and $G_2$, by Lemma 2.14, we can obtain that $G$ is equivalent to each of $G_1 + \Delta_{m_1}$ and $G_2 + \Delta_{m_2}$, where $m_i = |V(G_i)| - |V(G_i)|$ for $i = 1, 2$. Hence, we can conclude that $G_1 + \Delta_{m_1}$ and $G_2 + \Delta_{m_2}$ can be transformed into each other, via $G$.

**Outline of the proof of Theorem 2.7.** Let $F^2$ be a closed surface. By Theorem 2.12, $F^2$ admits finitely many irreducible triangulations up to homeomorphism, say $T_1, \ldots, T_n$. By
the above fact, there exists a positive integer \( N(F^2) \) such that \( T_1 + \Delta_{m_1}, \ldots, T_n + \Delta_{m_n} \) are equivalent to one another, where \( |V(T_i)| + m_i = N(F^2) \).

Let \( G_1 \) and \( G_2 \) be two triangulations on \( F^2 \) with \( |V(G_1)| = |V(G_2)| \geq N(F^2) \). Suppose that \( G_1 \) can be contractible to \( T_k \). Then \( G_1 \) is equivalent to \( T_k + \Delta_m \) with \( m \geq m_k \) since \( |V(G_1)| \geq N(F^2) \). Put \( l := m - m_k \). Then, \( T_k + \Delta_m \) is equivalent to \( T_1 + \Delta_{m_1+l} \) since \( T_1 + \Delta_{m_1} \) and \( T_k + \Delta_{m_k} \) are equivalent. Similarly, \( G_2 \) can be transformed into \( T_1 + \Delta_{m_1+l} \). Thus, since \( |V(G_1)| = |V(G_2)| \), \( G_1 \) and \( G_2 \) are equivalent, via the standard form \( T_1 + \Delta_{m_1+l} \). Therefore, the theorem holds.

By Theorems 2.1, 2.4, 2.5 and 2.6, the value of \( N(\cdot) \) can be determined for the sphere \( S_0 \), the torus \( S_1 \), the projective plane \( N_1 \) and the Klein bottle \( N_2 \) as follows:

\[
N(S_0) = 4 \quad N(S_1) = 7 \quad N(N_1) = 6 \quad N(N_2) = 8
\]

For each of these closed surfaces, the value \( N(\cdot) \) coincides with the number of vertices of the smallest triangulation on it since any two triangulations with the same number of vertices must be equivalent. However, this phenomenon does not hold in general. There are closed surfaces with high genus which admit inequivalent triangulations with the same number of vertices [63].

### 2.3 Quadrangulations

For any two bipartite quadrangulations on a closed surface, they are equivalent if they have the same and sufficiently large number of vertices. However, in general, there exists a pair of non-bipartite quadrangulations which are not equivalent even if they have the same and sufficiently large number of vertices. To explain the equivalence for non-bipartite quadrangulations, Nakamoto introduced an invariant for a quadrangulation, which is called a “cycle parity”. In this section, we shall introduce the results on the equivalence for (non-)bipartite quadrangulations.

#### 2.3.1 Diagonal slides and diagonal rotations

By the definition of diagonal transformations, there are two kinds of diagonal transformations in quadrangulations. A diagonal slide is a sliding an edge as shown in Figure 2.9, and a diagonal rotation is a rotating a path of length 2, where the middle vertex has degree 2 as shown in Figure 2.10.

![Figure 2.9: A diagonal slide](image-url)
A diagonal slide cannot be omitted since a diagonal rotation can be applied only for a vertex of degree 2, and for any closed surface $F^2$, there are infinitely many quadrangulations on $F^2$ with the minimum degree at least 3. Moreover, a diagonal rotation is also necessary since a diagonal slide preserves the bipartition size of a given bipartite quadrangulation. (That is, if two given bipartite quadrangulations have the different bipartition size, then we need a diagonal rotation to change the bipartition size.)

First, we introduce the following lemma. This is an essential fact to show the equivalence for two quadrangulations in the later argument.

**Lemma 2.15** A 2-vertex in a quadrangulation can be moved to a neighboring face by exactly two diagonal slides.

**Proof.** As shown in Figure 2.11, a 2-vertex can be moved to a neighboring face by exactly two diagonal slides. □

As mentioned above, a diagonal rotation is needed to transform a bipartite quadrangulation $G$ into another $G'$ if $G$ and $G'$ have the different bipartition size. On the other hand, for non-bipartite quadrangulations, we do not need a diagonal rotation as follows.

**Lemma 2.16** The diagonal rotation in a non-bipartite quadrangulation can be obtained by a sequence of diagonal slides.

**Proof.** Let $G$ be a non-bipartite quadrangulation on a closed surface $F^2$. Every 2-vertex $v$ can be moved to any face by Lemma 2.15, but we cannot control freely the position of the two edges incident with $v$ without diagonal rotations. However, moving $v$ along an
odd cycle takes the same effect as the diagonal rotation of $v$: We first move $v$ to a face $f$ incident with $C$ if $v$ is not incident with such a cycle $C$. Next, $v$ is again moved to $f$ along $C$, and then, we carry $v$ to the first position (after applying this, we obtain the same effect as the diagonal rotation of $v$ in $f$ since $|C|$ is odd). This operation is always applicable since $G$ has an odd cycle.

For any surface $F^2$ except the sphere, there exist both bipartite and non-bipartite quadrangulations on $F^2$. Since diagonal transformations in quadrangulations preserve the bipartiteness of quadrangulations, a bipartite quadrangulation cannot be transformed into a non-bipartite quadrangulation by diagonal transformations even if they have the same number of vertices. In the next subsection, we introduce results on bipartite quadrangulations, and then, we present the equivalence for non-bipartite quadrangulations in the third subsection.

### 2.3.2 Bipartite case

The bipartite quadrangulation shown in Figure 2.12 is a standard form $S_{m,n}$ which consists of the equator $abcd$ ($a$ and $c$ are black and $b$ and $d$ are white), $m - 2$ black 2-vertices in the northern hemisphere which are adjacent to $b$ and $d$, and $n - 2$ white 2-vertices in the southern hemisphere which are adjacent to $a$ and $c$.

![Figure 2.12: The standard form $S_{m,n}$](image)

We denote the quadrangulation obtained from $G$ by adding $S_{m,n}$ to a face of $G$ by $G + S_{m,n}$. Note that $G + S_{2,2} \cong G$. This notation $G + S_{m,n}$ seems to be ambiguous since we can make different quadrangulations by adding $S_{m,n}$ to different faces of $G$. However, we can show that they can be transformed into each other only by diagonal slides by Lemma 2.15. So, $G + S_{m,n}$ denotes a unique quadrangulation up to equivalence. Moreover, we can see that $G + S_{m,n} + S_{m',n'}$ can be transformed into $G + S_{m+m'-2,n+n'-2}$. (Note that $|V(G + S_{m,n})| = |V(G)| + (m - 2) + (n - 2)$.)

Then, by the similar method to Negami’s idea which is used to prove Theorem 2.7, Nakamoto [53] proved the following two theorems. (That is, Nakamoto [53] prepared a contraction in quadrangulations called a “face contraction”, and showed that the number of “irreducible quadrangulations” is finite.) The second theorem implies that if two given bipartite quadrangulations have the same bipartition size, then we do not need a diagonal rotation.

34
Theorem 2.17 (Nakamoto [53]) For any closed surface $F^2$, there exists a positive integer $M(F^2)$ such that any two bipartite quadrangulations $G$ and $G'$ on $F^2$ are equivalent if $|V(G)| = |V(G')| \geq M(F^2)$.

Theorem 2.18 (Nakamoto [53]) For any closed surface $F^2$, there exist positive integers $M_B(F^2)$ and $M_W(F^2)$ such that any two bipartite quadrangulations $G$ and $G'$ on $F^2$ can be transformed into each other only by diagonal slides if $|V_B(G)| = |V_B(G')| \geq M_B(F^2)$ and $|V_W(G)| = |V_W(G')| \geq M_W(F^2)$.

In Theorems 2.17 and 2.18, we cannot remove the condition that the number of vertices is sufficiently large from the statements. For example, we consider two quadrangulations $K_{4,4} + S_{2,3}$ and $K_{3,6}$ on the torus shown in the left hand and the right hand of Figure 2.13, respectively. (A unique white 2-vertex of $K_{4,4} + S_{2,3}$ is a (middle) vertex of $S_{2,3}$.)

![Figure 2.13: Quadrangulations $K_{4,4} + S_{2,3}$ and $K_{3,6}$ on the torus](image)

Note that any diagonal slide cannot be applied to each edge of $K_{3,6}$ since each black vertex is adjacent to each white one. Hence, although $K_{4,4} + S_{2,3}$ and $K_{3,6}$ have the same number of vertices, these quadrangulations cannot be equivalent. (Also, we can easily construct two quadrangulations with the same bipartition size which cannot be transformed into each other only by diagonal slides.)

2.3.3 Non-bipartite case

On the other hand, any two non-bipartite quadrangulations are not necessarily equivalent even if they have the same and sufficiently large number of vertices (for example, see [53]). Then, for non-bipartite quadrangulations, the equivalence can be described by a notion called a “cycle parity”. (In this thesis, we do not minutely explain a cycle parity. For the details, see [55].)

It is easy to see that any two cycles of a quadrangulation on a closed surface $F^2$ have the same length modulo 2 if they are homotopic to each other on $F^2$. Two quadrangulations $G_1$ and $G_2$ on a closed surface $F^2$ are said to have the same cycle parity if for each closed curve $l$ on $F^2$, a cycle $C_1$ of $G_1$ and a cycle $C_2$ of $G_2$ both of which are homotopic to $l$ on $F^2$ have the same length modulo 2. We also see that both of a diagonal slide and a diagonal rotation change the set of cycles of a quadrangulation on a closed surface $F^2$ but preserve the cycle parity of the graph. Hence, if two quadrangulations do not have the
same cycle parity, then they are not equivalent. Moreover, Nakamoto [52] showed that the converse of the above is also true as follows.

**Theorem 2.19 (Nakamoto [52])** For any closed surface $F^2$, there exists a positive integer $M'(F^2)$ such that any two quadrangulations $G$ and $G'$ on $F^2$ with $|V(G)| = |V(G')| \geq M'(F^2)$ are equivalent if they have the same cycle parity.

### 2.4 $N$-angulations for $N \geq 5$

In this section, we describe the results of equivalence for $N$-angulations on the sphere by diagonal transformations, where $N \geq 5$ is a positive integer. In particular, by the definition of diagonal transformations, those for pentangulations are the transformations $A$ and $B$ shown in Figure 2.14, and those for hexangulations are the transformations $A$, $B$ and $C$ shown in Figure 2.15.

![Figure 2.14: Transformations $A$ and $B$](image)

![Figure 2.15: Transformations $A$, $B$ and $C$](image)

For $N$-angulations, there is a big difference between the case when $N \leq 4$ and when $N \geq 5$. Let $G$ be an $N$-angulation on a closed surface $F^2$ and let $l$ be a path of length at least one shared by exactly two faces $f$ and $f'$ in $G$. If $N \leq 4$, $\partial(f \cup f')$ is always a cycle, where $f \cup f'$ denotes the subgraph of $G$ consisting of vertices and edges on the boundaries of $f$ and $f'$. On the other hand, when $N \geq 5$, $\partial(f \cup f')$ might not be a cycle. (For example, $u$ and $v$ might be the same in the leftmost of Figure 2.14.) Hence, for any integer $N \geq 5$, the argument of equivalence for $N$-angulations is more complicated than those of triangulations and quadrangulations. However, by checking carefully the situation around an edge (or a path) to which a diagonal transformation is applied, we proved the following theorems. Note that Theorem 2.20 is a corollary of the main theorem in [36] (which is Corollary 3.2 in Page 52). Therefore, the proof of Theorem 2.20 is not written in this chapter.

**Theorem 2.20** Any two pentangulations on the sphere with the same number of vertices are equivalent.
Theorem 2.21 Any two hexangulations on the sphere with the same number of vertices are equivalent.

In [45], it was conjectured that any two $N$-angulations on the sphere with the same number of vertices are equivalent. However, it seemed that the proof would be a routine with a case-by-case argument. Then, we develop a more general technique for proving the statement, and hence, we have the following.

Theorem 2.22 For any fixed integer $N \geq 7$, any two $N$-angulations on the sphere with the same number of vertices are equivalent.

By summarizing the results of equivalence for $N$-angulations, for any integer $N \geq 3$, every two $N$-angulations on the sphere with the same number of vertices are equivalent.

2.4.1 Necessity of the transformations

In this subsection, we shall describe the necessity of diagonal transformations in Theorems 2.21 and 2.22. (For Theorem 2.20, since we will describe the necessity of $A$ and $B$ in the next chapter, we now omit the explanation.)

First, we consider the necessity of transformations $A$, $B$ and $C$ which are diagonal transformations in hexangulations. The standard form of hexangulations on the sphere consists of paths of length 3 in which each middle vertex has degree exactly 2 (see Figure 2.16). Clearly, only $C$ can be applied to the graph. A 1-subdivided triangulation is a hexangulation on the sphere obtained from a plane triangulation $T$ by subdividing each edge of $T$ with a single vertex of degree 2 (see Figure 2.17) to which neither $A$ nor $C$ can be applied. There is a hexangulation obtained from a plane quadrangulation by adding a path of length four, where each middle vertex is a 2-vertex, into each face as a diagonal (see Figure 2.18) to which neither $B$ nor $C$ can be applied. Therefore, each of three transformations $A$, $B$ and $C$ cannot be omitted from Theorem 2.21.

![Figure 2.16: The standard form](image1)

![Figure 2.17: A 1-subdivided requiring $A$ triangulation](image2)

![Figure 2.18: A hexangulation](image3)

Next, we show that there are $N$-angulations on the sphere which require the specified transformations. By the definition of diagonal transformations, the length of the path, where the middle vertices have degree exactly 2, flipped by a diagonal transformation in $N$-angulations is from one to $\left\lfloor \frac{N}{2} \right\rfloor$ for any fixed $N \geq 7$. Then, a diagonal transformation
applicable to a path of length \( l \), where the middle vertices have degree exactly 2, is simply called an \( l \)-flip in this subsection. Now, we introduce \( N \)-angulations requiring an \( l \)-flip for each \( l \in \{1, 2, \ldots, \lceil \frac{N}{2} \rceil \} \).

There is an \( N \)-angulation on the sphere obtained from a plane quadrangulation with the minimum degree 3 by adding a path of length \( N - 2 \) into each face as a diagonal. Such \( N \)-angulations on the sphere clearly require 1-flips. For other integer \( k \leq \lceil \frac{N}{2} \rceil \), we construct an \( N \)-angulation on the sphere requiring a \( k \)-flip as follows:

Let \( k \) and \( m \) be positive integers, where \( 2 \leq k \leq \lceil \frac{N}{2} \rceil \) and \( m = N - k \). Prepare two vertices \( u \) and \( v \) which lie on the northpole and the southpole, respectively. Then, we alternately add a path of length \( k \) with the middle vertices of degree exactly 2 and a path of length \( m \) with the middle vertices of degree exactly 2 so that each path has the same ends \( u \) and \( v \) as shown in Figure 2.19. It is easy to see that such \( N \)-angulations require a \( k \)-flip since \( m \) is always larger than \( \lceil \frac{N}{2} \rceil \).

**Figure 2.19:** An \( N \)-angulation on the sphere requires a \( k \)-flip

### 2.4.2 Proof of Theorem 2.21

Before we prove the theorem, we show the following lemmas. Let \( xy \) be an edge in a hexangulation, and we suppose that \( xy \) can be flipped by Transformation A to join two vertices \( a \) and \( b \). In this case, we denote \( xy \to ab \). For Transformation B (resp., C) applied to a path \( xvy \) (resp., \( xvuy \)), we similarly denote \( xvy \to avb \) (resp., \( xvuy \to avub \)).

**Lemma 2.23** Let \( G \) be a hexangulation on the sphere and let \( x \in V(G) \) with \( \deg(x) \geq 3 \).

1. Let \( e = xy \) be an edge with \( \deg(y) \geq 3 \). Then \( e \) can be flipped by Transformation A to reduce the degree of \( x \).

2. Let \( P = xyz \) be a path of length 2 with \( \deg(y) = 2 \) and \( \deg(z) \geq 3 \). Then \( P \) can be flipped by Transformation B to reduce the degree of \( x \).

3. Let \( P = xyzw \) be a path of length 3 with \( \deg(y) = \deg(z) = 2 \) and \( \deg(w) \geq 3 \). Then \( P \) can be flipped by Transformation C to reduce the degree of \( x \).
Proof. (1) We apply Transformation A. If \( xy \) is shared by two faces \( xabcdy \) and \( xpqrsy \), there are two choices to move \( xy \), that is, \( xy \rightarrow as \) or \( xy \rightarrow pd \). By the planarity, one of those two transformations is possible without loss the simpleness and 2-connectedness of graphs. That is, if \( xy \rightarrow as \) is impossible, then \( G \) has an edge \( as \). In this case, the 4-cycle \( axys \) separates \( p \) and \( d \) in the interior and exterior. Hence we have \( xy \rightarrow pd \).

(2) We apply Transformation B. If \( xyz \) is shared by two faces \( xabczy \) and \( xpqrzy \), there are two choices to move \( xyz \), that is, \( xyz \rightarrow ayr \) or \( xyz \rightarrow pyc \). If both are impossible, we have both \( a = r \) and \( c = p \). Similarly to (1), it is impossible since \( a \neq c \) by the 2-connectedness of graphs.

(3) Transformation C is always possible without breaking the simpleness and 2-connectedness of graphs.

We define Transformation D as shown in Figure 2.20, and let us consider whether any two hexangulations with the same number of vertices can be transformed into each other by Transformations A, B, C and D. By the following lemma, this proves Theorem 2.21.

![Figure 2.20: Transformation D](image_url)

**Lemma 2.24** Let \( G \) be a hexangulation on the sphere and let \( x \in V(G) \) with \( \deg(x) \geq 3 \). Let \( P = xyzwv \) be a path of length 4 with \( \deg(y) = \deg(z) = \deg(w) = 2 \) and \( \deg(v) \geq 3 \). Then \( P \) can be flipped by Transformation D to reduce the degree of \( x \). Moreover, Transformation D can be derived from Transformations A, B, and C.

**Proof.** It is easy to see that Transformation D is always possible without breaking the simpleness and 2-connectedness of graphs.

We consider whether Transformation D can be obtained by a sequence of Transformations A, B, and C. Suppose that the union of two face sharing a path \( xyzwv \) of length four is bounded by a 4-cycle \( xavp \) as shown in Figure 2.20. We consider a face neighboring to the quadrilateral region \( \Gamma \) bounded by the 4-cycle \( xavp \). By the planarity, we can always find a face \( f \) such that the common edges of \( f \) and \( \Gamma \) induce a connected graph. Moreover, we can see that the number of such common edges is at most two by the simpleness and 2-connectedness of graphs, and hence we have the following.

**Case 1.** There is a face \( f \) sharing exactly one edge with \( \Gamma \).

Without loss of generality, we may suppose that \( f \) contains the edge \( pv \). Transformation D can be derived from Transformations A, B, and C, as shown in Figure 2.21. Since the intersection of \( f \) and \( \Gamma \) is only \( pv \), all transformations in Figure 2.21 preserve the simpleness and 2-connectedness of graphs.
Case 2. There is a face $f$ sharing exactly two edges with $\Gamma$.

Without loss of generality, we may suppose that $f$ contains the path $pva$. Transformation $D$ can be derived from Transformations $A$, $B$ and $C$, as shown in Figure 2.22. Since $f$ is bounded by a cycle, all transformations preserve the simpleness and 2-connectedness of graphs.

Therefore, Transformation $D$ can be derived from a sequence of Transformations $A$, $B$ and $C$ in all cases. ■
Now, we have prepared to prove Theorem 2.21. By Lemma 2.24, Theorem 2.21 is equivalent to Theorem 2.25, and hence we prove Theorem 2.25.

**Theorem 2.25** Any two hexangulations on the sphere with the same number of vertices can be transformed into each other by Transformations A, B, C and D, up to homeomorphism.

**Proof.** Let \( G \) be a hexangulation on the sphere with an outer cycle \( NxySzw \). By induction on \( |V(G)| \), we shall prove that \( G \) can be transformed into the standard form (shown in Figure 2.23), fixing the outer 6-cycle.

![Figure 2.23: The standard form in hexangulations on the sphere](image)

If \( G \) is isomorphic to a 6-cycle, then \( G \) can be regarded as the standard form, and hence we may suppose that \( |V(G)| > 6 \). For a face or a 2-cell region \( f \) of \( G \), let \( \partial f \) denote the boundary cycle of \( f \).

**Step 1.** We make \( \text{deg}(x) = \text{deg}(y) = 2 \).

First, applying Transformations A, B, C and D, we can make \( \text{deg}(x) = 2 \) by Lemmas 2.23 and 2.24. We may suppose that \( \text{deg}(y) \neq 2 \). Let \( F_1 \) be the finite face containing \( x \), where \( \partial F_1 \) is supposed to be \( yu hkN x \). Since \( \text{deg}(y) \neq 2 \), we have \( u \neq S \). So we let \( F_2 \) be a finite face sharing \( uy \) with \( F_1 \), where \( \partial F_2 = yuabcd \) (see the left in Figure 2.24).

Now, if \( d \neq S \), we can apply Transformations A, B, C and D to a path or an edge which contains \( yd \) without increasing \( \text{deg}(x) \) by Lemmas 2.23 and 2.24. By repeating this operation, we have \( d = S \), that is, \( \text{deg}(y) = 3 \) (see the right in Figure 2.24).

Let \( P \) be the path shared by \( \partial F_1 \) and \( \partial F_2 \), whose middle vertices are of degree 2 and whose end vertices are \( y \) and a vertex of degree 3. We consider the following four cases, according to the length of \( P \).

**Case 1.** \( P = yu \).

In the right hand of Figure 2.24, we consider whether Transformation A can be applied to \( yu \) to join \( h \) and \( S \). If this is applicable, then we are done. Otherwise, \( G \) has an edge \( hS \). In any case, \( yu \) can be switched to an edge \( ax \) by Transformation A, since a 4-cycle \( Sh uy \) separates \( a \) and \( x \) in the interior and exterior. Following this, we can switch \( ax \) to an edge \( bN \) by Transformation A since a 6-cycle \( Sh uaxy \) separates \( b \) and \( N \) in its interior and exterior. Thus, we can make \( \text{deg}(x) = \text{deg}(y) = 2 \).

41
Case 2. $P = yuv$ with $\deg(u) = 2$.

See Figure 2.25, which is obtained from the configuration in the right hand of Figure 2.24 by identifying $a$ with $h$. Consider Transformation $B$ to flip $yuv$ to join $k$ and $S$. If this is applicable, then we are done. Otherwise, we have $k = S$. In this case, we have $b \neq x$ and $c \neq N$ because a 4-cycle $Svuy$ separates $\{b, c\}$ and $\{x, N\}$ in the interior and exterior. Hence $yuv$ can be switched to a path $Nuc$ by applying Transformation $B$ twice. Thus, we can make $\deg(x) = \deg(y) = 2$.

Case 3. $P = yvt$ with $\deg(u) = \deg(v) = 2$.

See Figure 2.26, which is obtained from the configuration in Figure 2.25 by identifying $b$ with $k$. Consider Transformation $C$ to flip $yvt$ to join $S$ and $N$. By Lemma 2.23, we can obtain $\deg(x) = \deg(y) = 2$.

Case 4. $P = yvt.N$ with $\deg(u) = \deg(v) = \deg(t) = 2$.

See Figure 2.27, which is obtained from the configuration in Figure 2.26 by identifying $c$ with $N$. Consider Transformation $A$ to flip an edge $NS$ to make $\deg(t) = 3$. Then we apply Transformation $C$ to $yvt$ to join $N$ and $S$ as in Case 3.

Therefore we can make $\deg(x) = \deg(y) = 2$ in all cases.
Step 2. We make $\deg(N) \geq 3$ and $\deg(S) \geq 3$, keeping $\deg(x) = \deg(y) = 2$.

We have $\deg(x) = \deg(y) = 2$ since we did in Step 1. If we have $\deg(N) \geq 3$ and $\deg(S) \geq 3$, then we are done. If $\deg(N) = \deg(S) = 2$, then $G$ must be a 6-cycle, contrary to the assumption on $|V(G)| > 6$. Hence, by symmetry, we may suppose $\deg(N) = 2$ but $\deg(S) \geq 3$ (see Figure 2.28). Here we may suppose $\deg(w) \geq 4$ or $\deg(S) \geq 4$ or $\deg(u) \geq 3$. (For otherwise, i.e., if $\deg(w) = \deg(S) = 3$ and $\deg(u) = 2$, then $G$ would have a face whose boundary is not a cycle, a contradiction.) Moreover, if $\deg(w) \geq 4$, then we can make $\deg(w) = 3$ as in Step 1 by Lemmas 2.23 and 2.24. Thus we may suppose $\deg(u) \geq 3$ or $\deg(S) \geq 4$.

First we assume $\deg(u) \geq 3$. We consider Transformation $A$ to flip $wu$ since $\deg(w) = 3$ and $\deg(u) \geq 3$. Let $uwzabc$ be a face sharing the edge $uw$ with the face $NxySuw$. Then $uw$ can be switched to make an edge $Nc$ by Transformation $A$, since a 4-cycle $zuwS$ separates $c$ and $N$ in the interior and exterior. Hence we can make $\deg(N) \geq 3$ and $\deg(S) \geq 3$.

Second we may suppose $\deg(u) = 2$ and $\deg(S) \geq 4$. Let $Suwzab$ be a face sharing the path $Suw$ with the face $NxySuw$. Since $N \neq b$, then we have $wuS \rightarrow Nub$ by Transformation $B$.

Therefore, we can make $\deg(N) \geq 3$ and $\deg(S) \geq 3$.

Figure 2.29: Operation in Step 3 for getting $G'$ from $G$
Step 3. Let $G'$ be the hexangulation obtained from $G$ by removing the path $NxyS$, and apply the procedures in Steps 1 and 2 to $G'$. (Figure 2.29)

By the inductive hypothesis, we can transform $G'$ into a standard form, fixing the boundary cycle of $G'$. Hence $G$ can be transformed into the standard form by Transformations A, B, C and D. ■

2.4.3 Proof of Theorem 2.22

Firstly, we shall show the following lemmas. A $k$-path is a path of length $k$ $(1 \leq k \leq N - 2)$ in which each middle vertex has degree exactly 2 and the two ends have degree at least 3 (it is not difficult to see that if $k \geq N - 1$, then we have multiple edges or loops). Moreover, for any $k$-path $P$, $P$ is flippable if $P$ can be flipped by diagonal transformations.

**Lemma 2.26** Let $G$ be an $N$-angulation. Any $k$-path in $G$ can be flipped by diagonal transformations if $1 \leq k \leq \lceil \frac{N}{2} \rceil$.

*Proof.* For any integer $1 \leq k \leq \lceil \frac{N}{2} \rceil$, since there are two choices to flip $k$-path $P$ in $G$, it is not difficult to check that the lemma holds by Jordan Curve Theorem. ■

By Lemma 2.26 and the definition of diagonal transformations in $N$-angulations, we can flip only $k$-paths such that $k \leq \lceil \frac{N}{2} \rceil$. Then, we next show that we can flip any $k$-path by diagonal transformations for each $1 \leq k \leq N - 2$.

**Lemma 2.27** Let $G$ be an $N$-angulation. For any integer $1 \leq k \leq N - 2$, any $k$-path in $G$ can be flipped by a sequence of diagonal transformations.

*Proof.* By Lemma 2.26, it suffices to prove the lemma that we consider a $k$-path such that $\lceil \frac{N}{2} \rceil + 1 \leq k \leq N - 2$. Let $P = v_0v_1v_2\cdots v_{k-1}v_k$ be a $k$-path in $G$, where $k = \lceil \frac{N}{2} \rceil + l$ $(1 \leq l \leq N - 2 - \lceil \frac{N}{2} \rceil)$. Let $f$ be a face, where $\partial f = v_0v_1v_2\cdots v_{k-1}v_ku_mu_{m-1}\cdots u_2u_1$ $(m = N - k - 1$, see the left hand of Figure 2.30). By symmetry, we consider to flip $P$ on the clockwise as shown in Figure 2.30 by diagonal transformations. Let $P' = v_0u_1u_2\cdots u_j$ be a $j$-path on $\partial f$, where $j \leq N - k$ and $u_{N-k} = v_k$. Then, we consider the following two cases depending on $j$.

![Figure 2.30: Flipping P](image-url)
Lemma 2.28

Let \( G \) be a \( 2m \)-angulation. Then \( G \) can be transformed into a \( 2m \)-angulation with at least one \( m \)-path by diagonal transformations.
Hence, we may suppose that $G$ has no $m$-paths. We first suppose that $G$ has no $s$-paths, where $s \leq m - 1$. In this case, $G$ has an $r$-path $P_r = v_0v_1v_2 \cdots v_{r-1}v_r$, where $r \geq m + 1$, since $G$ has at least one vertex of degree 2 by Euler’s formula. Let $f$ be a face of $G$ in which $\partial f$ includes $P_r$. In this case, we obtain an $m$-path by diagonal transformations by flipping the edge $e = xv_r$ on the clockwise $r - m$ times, where $e$ is an edge on $\partial f$. (We now have $\deg(x) \geq 3$ since it is easy to see that if $\deg(x) = 2$, then $G$ has a $s'$-path on $\partial f$ such that $s' \leq m - 1$.) Hence, we may suppose that $G$ has a $k$-path $P = v_0v_1 \cdots v_{k-1}v_k$, where $k \leq m - 1$.

We transform $P$ into an $m$-path by increasing $|P|$ by diagonal transformations. If $\deg(v_k) \geq 4$, then it is not difficult to see that we can make $\deg(v_k) = 3$ by diagonal transformations preserving $P$ by Lemmas 2.26 and 2.27 (see Figure 2.31).

![Figure 2.31](image)

Figure 2.31: We can flip intermediate paths between $u_1$ and $u_2$ preserving $P$.

Hence we may suppose $\deg(v_k) = 3$. Let $u_1$ and $u_2$ be two neighbors of $v_k$ which are not $v_{k-1}$, and let $P' = v_ku_1 \cdots$ and $P'' = v_ku_2 \cdots$ be a $d$-path and a $d'$-path, respectively ($1 \leq d, d' \leq 2m - k$).

Firstly, we suppose that $|P'| \geq 2$ (or $|P''| \geq 2$). We consider flipping $P'$ to increase $\deg(u_2)$. If this operation is applicable preserving $\deg(v_i) = 2$ ($1 \leq i \leq k - 1$), then we can increase $|P|$ by one since we have $\deg(v_k) = 2$. Otherwise, we have $u_2 = x$ or $P'$ is a $(2m - k)$-path, where $x$ is the other end vertex of flipped $P'$. In the first case, since we have $\deg(u_2) \geq 3$, we can increase $|P|$ by one by flipping $P'$ on the clockwise $|P|$ times. In the other case, we can flip $P''$ to increase $\deg(u_1)$ since we now have $|P''| < |P'|$ (otherwise, that is, if $|P''| = |P'|$, then it contradicts the 2-connectedness since $v_0$ becomes a cut vertex). Hence, we can increase $|P|$ by one by flipping $P''$ on the clockwise once.

Next, we suppose that $|P'| = |P''| = 1$, that is, $\deg(u_1) \geq 3$ and $\deg(u_2) \geq 3$. Then we consider flipping $P'$ to increase $\deg(u_2)$. If this operation is applicable, then we can increase $|P|$ by one. Otherwise, we have an edge $u_2y$, where $y$ is the other end vertex of flipped $P'$. In this case, we can increase $|P|$ by one by flipping $P''$ to increase $\deg(u_1)$. Therefore, since we can increase $|P|$ by one in both cases by diagonal transformations, we can obtain an $m$-path $P$ by repeating the above operations. ⊗

**Lemma 2.29** Let $G$ be a $2m$-angulation. For any $2m$-angulation obtained from $G$ by adding an $m$-path $P$ to a face of $G$, $P$ can be moved to any other face of $G$ by diagonal transformations.

**Proof of Lemma 2.29.** Let $\Gamma$ be a $2m$-angular region whose $P$ in its interior and let $f$ be a face sharing a $k$-path $P'$ with $\Gamma$, where $1 \leq k \leq 2m - 2$, and let $u$ and $v$ be two ends of $P'$, where they appear in this order on the anti-clockwise of $\partial \Gamma$. (If $f$ shares several paths with $\Gamma$, then we regard the shortest one among those paths as $P'$.) We consider to
move $P$ to $f$ by diagonal transformations. Since $P$ is flippable, we may suppose that one end of $P$ is $u$. The following three cases arise.

Case 1. $k \geq m$

In this case, we can move $P$ to $f$ by retaking $\Gamma$ as shown in Figure 2.32. (Note that if $k = m$, then two ends of $P$ are $u$ and $v$.)

Case 2. $2 \leq k < m$

In this case, we first flip $P'$ on the clockwise $m - k$ times. Then, since we now have a $k$-path $P''$ on $P$ without $u$ as its ends, we can move $P$ to $f$ by flipping $P''$ on the clockwise $m - k$ times.

Case 3. $k = 1$

In this case, we first flip $P'$ on the clockwise $m - 2$ times. Next, since we now have a 2-path $P''$ on $P$ without $u$ as its ends, we flip $P''$ on the clockwise $m - 2$ times. Then we flip an $(m - 1)$-path which contains $P'$ on the clockwise once, and finally, since we now have a 1-path (an edge) which consists of one edge of $P''$ without $u$ as its ends, we can move $P$ to $f$ by flipping the 1-path on the clockwise once.

Therefore, since we can move $P$ to $f$ by diagonal transformations in each case, the lemma holds by repeating the above operations. \(\diamond\)

(ii) $N = 2m + 1$ ($m \geq 3$)
In this case, we make a special form $\gamma$ shown in Figure 2.33 by diagonal transformations since removing all inner $2m - 1$ vertices of $\gamma$ preserves that $G$ is an $(2m + 1)$-angulation, and similarly to Case (i), we define removing $\gamma$, adding $\gamma$ and moving $\gamma$. Then we shall show the following two lemmas.

Figure 2.33: The form $\gamma$ consists of two $m$-paths and an edge (in total, $2m - 1$ vertices and $2m + 1$ edges) and three vertices $u_1, u_2$ and $u_3$. Moreover, $\deg(u_i) \geq 3$ for each $i \in \{1, 2, 3\}$.

**Lemma 2.30** Let $G$ be a $(2m+1)$-angulation. Then $G$ can be transformed into a $(2m+1)$-angulation with at least one $\gamma$ by diagonal transformations.

**Proof of Lemma 2.30.** If $G$ has a $\gamma$, then we are done. Hence we may suppose that $G$ has no $\gamma$. By repeating the operations in Lemma 2.28, it is easy to check that we can obtain a $\{2m - 1 (= N - 2)\}$-path. Hence, we may suppose that $G$ has an $(N - 2)$-path $P = xv_1v_2 \cdots v_{N-4}v_{N-3}y$ and let $f$ and $f'$ be two faces sharing $P$, where $\partial f = uxv_1v_2 \cdots v_{N-4}v_{N-3}y$ and $\partial f' = wxv_1v_2 \cdots v_{N-4}v_{N-3}y$. Firstly, we make $\deg(u) = 2$ by diagonal transformations preserving $P$ (in this case, we flip paths except two edges $ux$ and $uy$). This operation is always applicable by Lemmas 2.26 and 2.27. Next, we make $\deg(w) = 2$ by diagonal transformations preserving $P$. In this operation, if a flip makes $\deg(u) = 3$, then the corresponding path $P'$ has $x$ (or $y$) as its ends since we now have $\deg(u) = 2$. In this case, since the length of $P'$ is at most $N - 2$ (otherwise, contradicts that $G$ is an $N$-angulation), a cycle consists of the path and $wx$ (or $wy$) does not bound a face. Hence, for any flip in the operation, there exists the choice (clockwise or anti-clockwise) such that the transformation preserves $\deg(u) = 2$.

By the above operations, we now have $\deg(u) = \deg(w) = 2$ and we have $\deg(x) \geq 4$ and $\deg(y) \geq 4$ (otherwise, it contradicts the 2-connectedness). Therefore, after flipping a path $xuy$ on the anti-clockwise $m$ times, we can obtain $\gamma$ by flipping a $m$-path which consists the vertices of $P$ on the anti-clockwise once (to make $\deg(u) = 3$). ♦

**Lemma 2.31** Let $G$ be a $(2m + 1)$-angulation. For any $(2m + 1)$-angulation obtained from $G$ by adding $\gamma$ to a face of $G$, the $\gamma$ can be moved to any other face of $G$ by diagonal transformations.
Proof of Lemma 2.31. Similarly to the proof of Lemma 2.29, we consider to move $\gamma$ to $f$ from $\Gamma$ by diagonal transformations, where $f$ and $\Gamma$ are defined similarly to those in Lemma 2.29. As shown in Figure 2.34, since we can rotate $\gamma$ by diagonal transformations, it suffices to prove the lemma that we only consider the case $f \cap \Gamma$ is an edge $e$.

Let $e'$ be a 1-path (i.e. an edge) contained in $\gamma$. Then we first flip $e$ on the clockwise $m - 1$ times. Next, we flip $e'$ on the anti-clockwise $m$ times. Finally, by flipping $e$ on the clockwise once, we can move $\gamma$ to $f$. Hence, the lemma holds by repeating the above operations. \(\diamondsuit\)

By combining Lemmas 2.30 and 2.31, it is not difficult to see that Case (ii) can be proved similarly to Case (i). Therefore, we complete the proof of Theorem 2.22. \(\blacksquare\)

### 2.4.4 Extension of the results

By the results introduced in this chapter, for any positive integer $N \geq 3$, any two $N$-angulations on the sphere with the same number of vertices are equivalent. Moreover, for triangulations and quadrangulations, the theorems are extended to any other surface. Hence, for any positive integer $N \geq 5$, we want to extend the theorems to $N$-angulations on other surfaces. However, the proof of the theorems for $N$-angulations on the sphere with $N \geq 5$ strongly depends on the planarity of the graphs since we often use the Jordan Curve Theorem. Hence, we cannot immediately extend our results to general surfaces, but we believe that the following holds. (We are also assured that the argument of a “cycle parity” is needed for $N$-angulations when $N$ is even.)
Conjecture 2.32 Let $G$ and $G'$ be $N$-angulations on a closed surface $F^2$ with the same and sufficiently large number of vertices, where $N \geq 3$. Then the following holds:
(i) If $N$ is odd, then $G$ and $G'$ are equivalent.
(ii) If $N$ is even, then $G$ and $G'$ are equivalent if they have the same cycle parity.
Chapter 3

Structure of transition diagrams

In this chapter, we describe structures of transition diagrams for \(N\)-angulations by diagonal transformations. A transition diagram for \(N\)-angulations by diagonal transformations is a graph in which each vertex represents an \(N\)-angulation with some number of vertices, and two vertices are adjacent in the diagram if and only if they are transformed into each other by a diagonal transformation.

3.1 Triangulations and quadrangulations

By Theorem 2.7, \(T_{N,F^2}\) is connected if \(N \geq N(F^2)\) which is the value in Theorem 2.7, where \(T_{N,F^2}\) is a transition diagram for triangulations on a closed surface \(F^2\) with \(N\) vertices by diagonal flips. Moreover, note that if \(N < N(F^2)\), then \(T_{N,F^2}\) might not be connected (for the details, see [39, 42]). Hence, we can see that the transition diagram of triangulations has the very simple structure.

For the transition diagram of quadrangulations, by the results in Section 2.3, we can make it similarly to triangulations. However, the structure is different from that of triangulations. Let \(D_{M,F^2}\) be a transition diagram for quadrangulations on \(F^2\) with \(M\) vertices. Let \(Q_{M,F^2}\) be a subgraph of \(D_{M,F^2}\) which consists of bipartite quadrangulations, and let \(Q_{B,W,F^2}\) be a subgraph of \(Q_{M,F^2}\) which consists of elements in \(Q_{M,F^2}\) with \(B\) black vertices and \(W\) white ones. Moreover, let \(Q'_{M,F^2} = D_{M,F^2} \setminus Q_{M,F^2}\), that is, \(Q_{M,F^2}\) consists of non-bipartite quadrangulations. Now \(D_{M,F^2}\) is disconnected since an element in \(Q_{M,F^2}\) and another in \(Q'_{M,F^2}\) cannot be transformed into each other by diagonal transformations. By Theorem 2.17 and Theorem 2.18, \(Q_{M,F^2}\) is connected and \(Q_{B,W,F^2}\) is connected only by diagonal slides, respectively, where \(M, B\) and \(W\) are sufficiently large. Note that if the number of vertices is small, then \(Q_{M,F^2}\) (or \(Q_{B,W,F^2}\)) is disconnected similarly to triangulations (cf. the argument in Section 2.3). Moreover, \(Q'_{M,F^2}\) is disconnected in general since there are several equivalent classes depending on a “cycle parity”.

3.2 Pentangulations

In this section, we shall introduce our result of the transition diagram for pentangulations on the sphere. Throughout this section, a pentangulation on the sphere is simply called
a pentangulation since we only deal with that on the sphere in this section.

At first, Kanno and Su who are our co-authors in [36] told us that any two pentangulations with the same number of vertices are equivalent. However, their proof is complicated and has many case-by-case arguments. Moreover, they did not exactly describe pentangulations which require $A$ and $B$, respectively (for two transformations, see Figure 2.14). Hence, we first consider pentangulations require those transformations.

The dodecahedron shown in Figure 3.1 (or Figure 1.4) is a pentangulation. Since the graph has no 2-vertex, it requires only $A$. Moreover, we can obtain another pentangulation with the minimum degree 3 from the dodecahedron $D$ on the sphere by adding another dodecahedron $D'$ to a face $f$ of $D$ (that is, we identify an outer face of $D'$ and $\partial f$). Therefore, there are infinitely many pentangulations requiring only $A$.

For the transformation $B$, there is a pentangulation requiring it shown in Figure 3.2, which is called the standard form of pentangulations. The standard form can be constructed as follows: Put two vertices $u$ and $v$ on the sphere. Then, we alternately add paths of length 2 and 3 with the middle vertices of degree exactly 2 such that the ends are $u$ and $v$. (Note that the smallest standard form is a 5-cycle.) However, we cannot find pentangulations requiring $B$ other than the standard form. Hence, we guess that the standard form of pentangulations is a unique pentangulation requiring $B$, and then, we prove the following theorem, which includes Corollary 3.2 (this corollary is a first result by Kanno and Su). By the above argument, it is clear that each of $A$ and $B$ cannot be omitted from Theorem 3.1 and Corollary 3.2.

**Theorem 3.1** Let $G$ and $G'$ be pentangulations with the same number of vertices. Then $G$ and $G'$ can be transformed into each other by $A$ and $B$. In particular, if $G$ and $G'$ are not isomorphic to the standard form, then we do not need $B$.

**Corollary 3.2** Any two pentangulations with the same number of vertices are equivalent.

Now, let us consider the transition diagram of pentangulations. Let $\mathcal{P}_N$ be a transition diagram of pentangulations with $N$ vertices. By Corollary 3.2, $\mathcal{P}_N$ is connected. Moreover, by Theorem 3.1, $\mathcal{P}_N - \{S_p\}$ is connected only by $A$, where $S_p$ is the standard form. In particular, the degree of $S_p$ is exactly one in $\mathcal{P}_N$ and the incident edge to $S_p$ implies $B$. The transition diagram can be drawn as shown in Figure 3.3.
3.2.1 The structure $\mathcal{X}$ in pentangulations

In this subsection, we consider the special local structure $\mathcal{X}$ in pentangulations on the sphere, which is a subgraph included in a pentagonal non-facial region. Let $R = v_1v_2v_3v_4v_5$ be the pentagonal non-facial region, where $v_i \neq v_j$ if $i \neq j$ for every $i, j \in \{1, 2, 3, 4, 5\}$. The structure $\mathcal{X}$ consists of three vertices $a, b$ and $c$ inside $R$, three vertices $v_1, v_3$ and $v_4$ on the boundary of $R$ and edges $av_1, ab, ac, bv_3, cv_4$, where $\deg(a) = 3, \deg(b) = \deg(c) = 2$ and $\deg(v_k) \geq 3$ for each $k \in \{1, 3, 4\}$ (see Figure 3.4). Moreover, we denote $\mathcal{X}$ by $\mathcal{X}(u_1u_2u_3u_4u_5u_6)$, where $u_1, u_2, u_3, u_4, u_5$ and $u_6$ correspond to $a, b, c, v_1, v_3$ and $v_4$ in Figure 3.4, respectively.

Now we consider the following two operations. Adding $\mathcal{X}$ means that we add three vertices $a, b$ and $c$ to the interior of a pentagonal face $v_1v_2v_3v_4v_5$ and add edges $ab, ac, av_1, bv_3$ and $cv_4$ as shown in Figure 3.5, and Removing $\mathcal{X}$ is the inverse operation of adding $\mathcal{X}$ as shown in Figure 3.5.
It is easy to see that adding (or removing) \( \mathcal{X} \) preserves that the graph is a pentangulation on the sphere since all \( v_i \)'s are distinct. Moreover, for adding \( \mathcal{X} \), there exist five possibilities for the neighbor \( v_i \) of \( a \) in the interior of the 5-cycle shown in Figure 3.5, but Figure 3.6 shows that all positions can be regarded as the same up to \( \mathcal{A} \). In this case, we note that the operation shown in Figure 3.6 does not break the simpleness and the 2-connectedness.

In the end of this subsection, we shall prove the following lemma.

**Lemma 3.3** Let \( G \) be a pentangulation on the sphere. Any two pentangulations on the sphere obtained from \( G \) by adding \( \mathcal{X} \) to each can be transformed into each other by a sequence of only \( \mathcal{A} \).

**Proof.** It suffices to prove that we can move \( \mathcal{X} \) in a face of \( G \) into any other face of \( G \) only by \( \mathcal{A} \) (note that two obtained graph from \( G \) by adding \( \mathcal{A} \) have the same number of vertices). Let \( \Gamma = e_1e_2e_3e_4e_5 \) be a pentagonal region which has three inner vertices of \( \mathcal{X} \) in the interior, where \( e_i \) is an edge for each \( i \in \{1, 2, 3, 4, 5\} \). Let \( f \) be a neighboring face of \( \Gamma \) and, without loss of generality, we may suppose that \( e_1 \in f \cap \Gamma \) and \( e_5 \notin f \cap \Gamma \). In this case, we can move \( \mathcal{X} \) to \( f \) only by \( \mathcal{A} \) as shown in Figure 3.7. Therefore, by repeating the operations shown in Figures 3.6 and 3.7, \( \mathcal{X} \) can be moved to any other face of \( G \) only by \( \mathcal{A} \). ■

**3.2.2 Proof of Theorem 3.1**

In this subsection, we shall prove Theorem 3.1. We first prove the following. Let \( xy \) be an edge of a pentangulation on the sphere and we suppose that \( xy \) can be flipped by \( \mathcal{A} \).
to join two vertices $a$ and $b$. In this case, applying $A$ to $xy$ to join $a$ and $b$ is denoted by $xy \rightarrow ab$.

**Lemma 3.4** Suppose that $xy$ is an edge of a pentangulation $G$ on the sphere such that $\deg(x) \geq 3$ and $\deg(y) \geq 3$. Let $xuv_1u_2v_3$ and $xvy_1v_2v_3$ be the faces sharing the edge $xy$. If the operation $xy \rightarrow v_3u_1$ cannot be applied, then one of the following situations occurs:

1. $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} = \emptyset$ and $u_1v_3 \in E(G)$
2. $u_1 = v_2$
3. $u_2 = v_3$
4. $u_1 = v_3$

**Proof.** If $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} = \emptyset$, then since the operation $xy \rightarrow u_1v_4$ is applied unless $u_1v_3 \in E(G)$, we have the situation (1). Hence, we may suppose that $\{u_1, u_2, u_3\} \cap \{v_1, v_2, v_3\} \neq \emptyset$, and may also suppose that $v_2 \neq \{u_2, u_3\}$ and $v_1 \neq \{u_2, u_3\}$ since it is easy to see that $xy \rightarrow u_1v_3$ can be applied in those cases. Moreover, we now have $v_1 \neq u_1$ and $v_3 \neq u_3$ since $\deg(x) \geq 3$ and $\deg(y) \geq 3$ (otherwise, $G$ has multiple edges). Hence, the remainders of combinations are $u_1 = v_2$, $u_2 = v_3$ and $u_1 = v_3$. Then, since it is not difficult to check that every two of the three cases cannot simultaneously occur, the lemma holds. ■

By the above lemma, we can immediately obtain the following lemma since if $G$ has one of the four situations in Lemma 3.4, $G$ does not have them on the opposite side (for example, $G$ does not simultaneously have two edges $u_1v_3$ and $u_3v_1$ in Lemma 3.4 by the planarity).

**Lemma 3.5** Suppose that $xy$ is an edge of a pentangulation on the sphere such that $\deg(x) \geq 3$ and $\deg(y) \geq 3$. Let $xuv_1u_2v_3$ and $xvy_1v_2v_3$ be the faces sharing the edge $xy$. Then, one of the operations $xy \rightarrow v_3u_1$ or $xy \rightarrow u_3v_1$ can be applied. ■

Next, we show the following two lemmas. The second lemma (Lemma 3.7) is essential to prove Theorem 3.1.

**Lemma 3.6** Let $G$ be a pentangulation on the sphere with $|V(G)| \geq 8$ which is not the standard form. Then we can obtain a face $f = u_1u_2u_3u_4u_5$ such that $\deg(u_1) \geq 3, \deg(u_2) \geq 3$ and $\deg(u_3) = 2$ by applying $A$ to $G$ at most once.

**Proof.** Let $G$ have no face required in the lemma, and let $\delta(G)$ be the minimum degree of $G$. If $\delta(G) \geq 3$, then there exists a vertex $v$ of degree 3 in $G$ since $\frac{2|E(G)|}{|V(G)|} = \frac{10}{3} - \frac{20}{3|V(G)|}$ by Euler’s formula. Then, by applying $A$ to an edge incident to $v$, we can obtain the required face by Lemma 3.5. Hence, we may suppose that $\delta(G) = 2$. Let $u$ be a vertex of degree two and let $x$ and $y$ be the neighbors of $u$. By the simpleness, we may suppose $\deg(x) \geq 3$ up to symmetry. (Otherwise, that is, if $\deg(x) = \deg(y) = 2$, then there exist multiple edges by 2-connectedness.) Let $f = uv_1v_2y$ be a face, and we now have $\deg(v_1) = 2$, otherwise...
$G$ has a required face. Moreover, we may suppose that $\deg(y) \geq 3$ and $\deg(v_2) = 2$ up to symmetry. (Otherwise, it is easy to see that $f$ is a required face or $G$ is not simple.) Let $f' = xv_1v_2yv_3$ be a face sharing a path $xv_1v_2y$ with $f$. We now have $\deg(v_3) = 2$, otherwise $f'$ is a required face. Then, we consider the next face $f'' = xv_4v_5yv_3$ sharing a path $xv_3y$ with $f'$ similarly to $f$. By repeating the above argument, it is easy to see that $G$ is the standard form, which is a contradiction.

Lemma 3.7 Let $G$ be a pentangulation on the sphere which is not the standard form. Then $G$ can be transformed into a pentangulation on the sphere with at least one $X$ only by $A$.

Proof. If $G$ already has $X$, then we are done. Hence, we may suppose that $G$ has no $X$ and $|V(G)| \geq 8$ (otherwise, $G$ is the standard form with five vertices by Euler’s formula). By Lemma 3.6, $G$ has a face $f = xyzv_1v_2$ such that $\deg(x) \geq 3$, $\deg(y) \geq 3$ and $\deg(z) = 2$ by applying $A$ once. Then we consider the following steps for surroundings of $f$.

Step 1. Make $\deg(y) = 3$

Suppose that $\deg(y) \geq 4$. Let $f' = xyua_1a_2$ be a face sharing $xy$ with $f$, and let $f'' = yub_1b_2b_3$ be a face sharing $yu$ with $f'$ (see Figure 3.8). We consider to reduce $\deg(y)$ by $A$.

Case 1. $\deg(u) \geq 3$ (Figure 3.8)

We can always reduce $\deg(y)$ by applying $A$ to $yu$ by Lemma 3.5.

Case 2. $\deg(u) = 2$ (Figure 3.9: This configuration is obtained from Figure 3.8 by identifying $a_1$ and $b_1$.)

After applying $A$ to $xy$ to make $\deg(u) = 3$, we replace $u$ as $x$. If this operation is applicable, we can reduce $\deg(y)$. (Note that this operation is also applicable if $a_2 = b_2$.) Otherwise, by Lemma 3.4, we have $u' = v_2$ since $u \not= \{v_1, v_2\}$. In this case, if $\deg(a_2) \geq 3$, then after applying $A$ to $v_2a_2$ to make $\deg(u) = 3$, we reduce $\deg(y)$ similarly to Case 1.

Hence, we suppose that $\deg(a_2) = 2$. Note that we now have $\deg(x) \geq 4$ (otherwise, this contradicts that $G$ is 2-connected), and let $r \not= v_2, a_2$ be the next vertex of $v_2$ on the clockwise rotation of $x$. After $v_2x \rightarrow v_1r$, we can reduce $\deg(y)$ by $xy \rightarrow ru$. These
operations can be applied since the 3-cycle $xa_2v_2$ separates $v_1, u$ and $r$. Following this, we replace $r$ and $u$ as $v_2$ and $x$, respectively.

For the operations in each case, note that we preserve $\deg(x) \geq 3$ and $\deg(z) = 2$. Therefore, since we can reduce $\deg(y)$ preserving $\deg(x) \geq 3$ and $\deg(z) = 2$ in each case, we suppose that $\deg(z) = 2$ and $\deg(y) = 3$, and we consider the next step.

**Step 2.** Make $\deg(x) = 3$

Let $f' = x'yzv_1v_2'$ be the face sharing a path $yzv_1$ with $f$. Also, we let $f_1 = xyx'r_2r_1$ be the face sharing a path $xyx'$ with $f \cup f'$, and let $f_2 = xr_1a_2a_1u$ be the face sharing an edge $xr_1$ with $f_1$. Now, if $u = v_2$, then we already have $\deg(x) = 3$. Hence, we may assume that $u \neq v_2$ and $\deg(x) \geq 4$. We consider the following two cases to reduce $\deg(x)$ only by $A$.

Case 1. $\deg(r_1) \geq 3$

If we can apply $xr_1 \rightarrow ur_2$, then we are done. Hence, we suppose that the operation is not applicable, that is, one of (1) $u = r_2$, (2) $u = x'$, (3) $a_1 = r_2$ or (4) $\{u, a_1, a_2\} \cap \{x', r_2\} = \emptyset$ and an edge $ur_2$ occurs by Lemma 3.4. In each case of them, we can apply $xr_1 \rightarrow ya_2 \rightarrow x'a_2$ since $a_1 \neq x'$. Note that if $a_1 = x'$, then we can apply $xr_1 \rightarrow ur_2$, and hence, we can always decrease $\deg(x)$ in this case.

Case 2. $\deg(r_1) = 2$ ($a_2 = r_2$)

After applying $xy \rightarrow r_1z, v_1z \rightarrow v_2y$, we replace $x', r_2, y$ and $r_1$ as $v_2', x', v_1$ and $y$, respectively. If this operation is applicable, then we are done. Otherwise, we have $v_2 = x'$. In this case, after applying $yx' \rightarrow zr_2 \rightarrow v_1r_1$ and $xr_1 \rightarrow yr_2$, we replace $r_1$ and $r_2$ as $v_2'$ and $x', r_1$ respectively. (Note that this operation is now applicable since the 3-cycle $x'yx$ separates $v_1$ and $a_2(= r_2)$ in the interior and the exterior.)

Therefore, we can reduce $\deg(x)$ only by $A$, which preserves $\deg(z) = 2$ and $\deg(y) = 3$ in both cases. Hence, we suppose that $\deg(z) = 2$ and $\deg(x) = \deg(y) = 3$ (see Figure 3.11), and we consider the final step.

**Step 3.** Make $X$
If \( \deg(x') = 2 \), \( \deg(v_1) \geq 3 \) and \( \deg(v_2') \geq 3 \), then it is easy to see that we already have \( \mathcal{X}(yx'zyv_2'v_1) \). Therefore, we may suppose that \( \deg(x') \geq 3 \) or \( \deg(v_1) = 2 \) or \( \deg(v_2') = 2 \), and hence, it suffices to consider the following cases.

Case 1. \( \deg(x') \geq 3 \) and \( \deg(v_1) \geq 3 \)

If \( \deg(v_2) = \deg(r_1) = 2 \) (\( a_1 = v_1 \) and \( a_2 = r_2 \)), then we already have \( \mathcal{X}(xv_2r_1yv_1r_2) \) since \( \deg(r_2) \geq 3 \) (otherwise, the six vertices \( v_1, v_2, x, r_1, r_2 \) and \( x' \) lie on the boundary of one face, which contradicts that \( G \) is a pentangulation on the sphere). Hence, we may assume that \( \deg(v_2) \geq 3 \) or \( \deg(r_1) \geq 3 \). In this case, we can obtain \( \mathcal{X}(yzxx'v_1r_1) \) (resp., \( \mathcal{X}(yxx'v_1v_2) \)) if \( v_2x \to v_1r_1 \) (resp., \( r_1x \to r_2v_2 \)) is applicable. If \( v_2x \to v_1r_1 \) (resp., \( r_1x \to r_2v_2 \)) is not applicable, then \( v_1 = r_1, v_1 = a_2 \) or \( v_1r_1 \in E(G) \) and \( v_1 \neq \{r_1, a_2\} \) (resp., \( v_2 = v_2, r_2 = a_1, v_2 = x' \) or \( v_1r_1 \in E(G) \) and \( r_2 \neq \{v_2, a_1\} \)) by Lemma 3.4. However, in each of the three (resp., four) case, we can apply the other operation by Jordan Curve Theorem (for example, if \( v_1 = r_1 \), then the 4-cycle \( r_1xyz \) separates \( \{r_2, x'\} \) and \( \{v_2, a_1\} \) in the interior and the exterior).

Case 2. \( \deg(x') \geq 3 \) and \( \deg(v_1) = 2 \)

We now have \( v_2 = v_2' \) and \( \deg(v_2') \geq 3 \) (note that if \( \deg(v_2') = 2 \), then we have \( x = x' \), which contradicts to the simplicity of \( G \)). Hence, we can obtain \( \mathcal{X}(zyv_1r_1x'v_2') \) by \( xy \to r_1z \).

Case 3. \( \deg(x') = 2 \) and \( \deg(v_1) = 2 \)

Now, since \( v_2' = r_2 = v_2 \) and \( x, v_1, x', a_1 \) and \( r_1 \) are clearly distinct, we have \( \deg(v_2') \geq 5 \), and we also have \( \deg(r_1) \geq 4 \) (otherwise, \( G \) has a cut vertex \( v_2' \) since the neighbors of \( x \) and \( r_1 \) are in the set \( \{v_2', r_1, x, y, z, v_1, x'\} \) and the 3-cycle \( v_2'x_1r_1 \) separates \( y, z, x', v_1 \) and other vertices of \( G \) in the interior and the exterior, which is a contradiction). Hence we can obtain \( \mathcal{X}(zyv_1r_1x'v_2') \) by flipping \( v_2r_1 \) to make \( \deg(x') = 3 \) and \( xy \to r_1z \).

Case 4. \( \deg(x') = 2 \), \( \deg(v_1) \geq 3 \) and \( \deg(v_2') = 2 \) (\( v_2' = r_2 \) and \( v_1 = r_1 \))

We now have \( \deg(v_1) \geq 4 \) and \( \deg(v_2') \geq 4 \) since \( \deg(x) = 3 \) (otherwise, this contradicts \( G \) is a pentangulation on the sphere). Hence we can obtain \( \mathcal{X}(x'v_2yv_2v_1z) \) by flipping \( v_1v_2 \) to make \( \deg(z) = 3 \) and \( xy \to v_2x' \).
Hence, since we can make $X$ only by $A$ in all cases of Step 3, the lemma holds.

We have prepared to prove Theorem 3.1.

**Proof of Theorem 3.1.** Let $G$ and $G'$ be pentangulations on the sphere with the same number of vertices and let $G$ and $G'$ be not isomorphic to the standard form. Since $G$ and $G'$ are the same if $|V(G)| = 5$, we may suppose that $|V(G)| \geq 8$. By induction on $|V(G)|$, we shall prove that $G$ and $G'$ can be transformed into each other only by $A$.

By Lemma 3.7, $G$ and $G'$ can be transformed into pentangulations on the sphere with at least one $X$. Let $X$ and $X'$ be two $X$'s in $G$ and $G'$, respectively, and let $H$ (resp., $H'$) be a pentangulation on the sphere obtained from $G$ (resp., $G'$) by removing $X$ (resp., $X'$). If $H$ and $H'$ are not isomorphic to the standard form, then they can be transformed into each other only by $A$ by the inductive hypothesis. Hence, $G$ and $G'$ can be transformed into each other only by $A$ by Lemma 3.3 since they are now obtained from the same graph by adding $X$.

Now, we may suppose that $H$ (or $H'$) is isomorphic to the standard form. In this case, we can see that the transformation $B$ in $G$ (or $G'$) can be obtained from a sequence of $A$ by using $X$ (or $X'$) as shown in Figure 3.12. Therefore, since we can transform $H$ (or $H'$) into a pentangulation on the sphere which is not the standard form only by $A$, Theorem 3.1 holds by Lemma 3.3. (Note that $B$ is necessary at most once only when either $G$ or $G'$ is the standard form.)

![Figure 3.12: Applying B and A twice](image)

**3.3 Hexangulations**

In this section, we introduce our results on transition diagrams for hexangulations on the sphere, for which we have studied from the master’s course to the present. Since we only deal with hexangulations on the sphere in this section, let a *hexangulation* mean that on the sphere in this section (in the final subsection, we also consider hexangulations on
a closed surface other than the sphere). Moreover, in the final subsection, we consider problems on diagonal transformations in hexangulations on closed surfaces.

In the proof of Theorem 2.21, it was proved that any hexangulation can be transformed into the standard form by $A$, $B$ and $C$ shown in Figure 2.15. Recall that each of $A$, $B$ and $C$ cannot be omitted in Theorem 2.21, as follows (for the corresponding figures, see Page 37). Indeed if we apply one of $A$, $B$ and $C$ to a path $P$ in a hexangulation $G$, then the two end-vertices of $P$ must have degree at least three. (Otherwise, the resulting graph has a 1-vertex, which contradicts the 2-connectedness of the resulting graph.)

1-1) $C$ cannot be omitted since neither $A$ nor $B$ can be applied to the standard from,

1-2) $B$ cannot be omitted since neither $A$ nor $C$ can be applied to a 1-subdivided triangulation, i.e., a hexangulation obtained from a plane triangulation (which contains a complete graph $K_3$) by subdividing each edge with a single 2-vertex, and

1-3) $A$ cannot be omitted since there is a hexangulation obtained from a plane quadrangulation by adding a path of length four into each face as a diagonal, to which neither $B$ nor $C$ can be applied.

Observe that every hexangulation is bipartite, and let $\mathcal{H}_{m,n}$ denote the set of hexangulations with bipartition size $(m, n)$, where $m, n \geq 3$. Then, we consider whether two hexangulations in $\mathcal{H}_{m,n}$ can be transformed preserving the bipartition size similarly to quadrangulations. We note that $A$ and $C$ preserve the bipartition size of the graph, but $B$ does not. Hence we consider:

**Problem 3.8** Can any two hexangulations with the same bipartition size be transformed into each other only by $A$ and $C$ preserving the bipartition size?

However, this does not hold by (1-2), since a 1-subdivided triangulation admits only $B$. Such a hexangulation belongs to $\mathcal{H}_{k,3k-6}$ (or $\mathcal{H}_{3k-6,k}$), since a plane triangulation with $k$ vertices has exactly $3k - 6$ edges. Hence we have:

2-1) If $G, G' \in \mathcal{H}_{k,3k-6}$ but $G \neq G'$, then $G$ and $G'$ cannot be transformed only by $A$ and $C$.

Moreover, by (1-1), the standard form $S$ admits only $C$, and $S \in \mathcal{H}_{k,k}$. Hence we have

2-2) If $G, G' \in \mathcal{H}_{k,k}$ but $G \neq G'$, then $G$ and $G'$ cannot always be transformed only by $A$.

There are infinitely many hexangulations to which neither $B$ nor $C$ can be applied, however, we cannot find a hexangulation requiring only $C$ other than the standard form. Hence, we conjectured that almost no transformation $C$ is needed, and then, we prove the following theorem. Surprisingly, we do not need $C$ to transform an element in $\mathcal{D}_{m,n}$ with $m \neq 3n - 6$ and $m \neq n$ into another in the same set.

**Theorem 3.9** Let $\mathcal{H}_{m,n}$ be the set of all hexangulations with bipartition size $(m, n)$, where $m \geq n$. Then
(3-0) \( H_{m,n} \) is empty if \( m > 3n - 6 \),

(3-1) no two distinct hexangulations in \( H_{3n-6,n} \) can be transformed by \( \Delta \) or \( \Theta \),

(3-2) if \( m \neq 3n - 6 \), any two hexangulations in \( H_{m,n} \setminus \{ S \} \) can be transformed only by \( \Delta \), where \( S \) is the standard form.

In Theorem 3.9, (3-2) claims that the converse of (2-1) holds. Moreover, this also claims that the converse of (2-2) holds, that is, only the standard form requires \( \Theta \) in all hexangulations.

Observe that for any even number \( N \geq 6 \), there exists a hexangulation with exactly \( N \) vertices. (For example, consider the standard form with \( N \) vertices.) Note that if \( N \equiv 0 \) (mod 4), then (3-1) does not happen. Therefore, if \( N \equiv 0 \) (mod 4), the answer for Problem 3.8 is Yes, but we have NO in general by (2-1).

Now, we consider hexangulations in the subset \( H_{3n-6,n} \) (or \( H_{m,3m-6} \)). Observe that every 1-subdivided triangulation has \( 3n - 6 \) black (resp., white) vertices and \( n \) white (resp., black) vertices since every triangulation on the sphere with \( n \) vertices has exactly \( 3n - 6 \) edges by Euler’s formula (this formula means \( V - E + F = 2 \), where \( V, E \) and \( F \) are the number of vertices, edges and faces, respectively). In fact, this number bounds the balance of the number of black vertices and white ones as follows.

**Proposition 3.10** \( G \in H_{3n-6,n} \) if and only if \( G \) is a 1-subdivided triangulation.

Summarizing all facts listed above, we can make the transition diagram \( D_N \) of hexangulations with \( N \) vertices as shown in Figure 3.13. In the diagram, each vertex represents a hexangulation with \( N \) vertices, and every two vertices are adjacent in \( D_N \) if and only if they are transformed either \( \Delta \), \( \Theta \) or \( \Theta \). Then \( D_N \) must be connected by Theorem 2.21.

Let \( D_{m,n} \subset D_N \) denote the subgraph induced by the hexangulations with bipartition size \((m, n)\), where \( m + n = N \). By Theorem 3.9, we get the diagram as shown in Figure 3.13: By (3-2), if \( m \neq 3n - 6 \) and \( n \neq 3m - 6 \), \( D_{m,n} \) is connected by \( \Delta \), and \( \Theta \) is necessary only to transform the standard form into another hexangulation. By Proposition 3.10, if \( m = 3n - 6 \) (or \( n = 3m - 6 \)), the subset \( D_{m,n} \) consists of isolated vertices since any element in \( D_{m,n} \) requires \( \Theta \). Moreover, \( \Theta \) corresponds to edges of \( D_N \) joining \( D_{m,n} \) and \( D_{m-1,n+1} \), or \( D_{m,n} \) and \( D_{m+1,n-1} \).

### 3.3.1 1-subdivided triangulations

In this subsection, we prove Proposition 3.10, (3-0) and (3-1), and describe \( D_{3m-6,n} \) (or \( D_{m,3m-6} \)). A multi-triangulation is a triangulation that might have multiple edges.

**Proof of Proposition 3.10.** Let \( G \) be a hexangulation in \( H_{m,n} \) with \( m \geq n \). Observe \( |V(G)| = m + n \), and \( 6|F(G)| = 2|E(G)| \) since \( G \) is a hexangulation. Moreover, \( 2m \leq |E(G)| \) by 2-connectedness. Then, by Euler’s formula, we have

\[
m \leq 3n - 6.
\]
In the above inequality, if \( m = 3n - 6 \), then we have \( 2m = |E(G)| \). This implies that each black vertex of \( G \) has degree exactly 2. Hence \( G \) can be transformed into a plane multi-triangulation by smoothing black vertices (see Figure 3.14). Therefore, the proposition holds.

Proof of (3-0) and (3-1). The statement (3-0) holds by the inequality \( m \leq 3n - 6 \) in the proof of Proposition 3.10. Moreover, by Proposition 3.10 and (1-2), it is clear that (3-1) holds.

Finally, we introduce \( \mathbb{B}^2 \) as the application of \( \mathbb{B} \) twice as shown in Figure 3.15, which clearly preserves the bipartition size. Moreover, it is easy to see that \( \mathbb{B}^2 \) transforms a hexangulation in \( \mathcal{H}_{3n-6,n} \) (resp., \( \mathcal{H}_{m,3m-6} \)) into another in the same set since it preserves that every black (resp., white) vertex has degree exactly 2. Hence, we can make the theorem for the transition in \( \mathcal{H}_{3n-6,n} \) as follows.
Proposition 3.11 Any two hexangulations in $\mathcal{H}_{3n-6,n}$ can be transformed only by $\mathbb{B}^2$, through $\mathcal{H}_{3n-6,n}$.

Proof of Proposition 3.11. Let $G$ and $G'$ be hexangulations in $\mathcal{H}_{3n-6,n}$. Then, by Proposition 3.10, there exist two plane multi-triangulations $T$ and $T'$ which are obtained from $G$ and $G'$ by smoothing all black vertices of degree 2, respectively. As shown in Figure 3.16, we can see that a diagonal flip in $T$ corresponds to $\mathbb{B}^2$ in $G$. Moreover, for multi-triangulations, Negami and Watanabe [67] proved that any two plane multi-triangulations with the same number of vertices can be transformed into each other by diagonal flips. Therefore, since $T$ and $T'$ can be transformed into each other by diagonal flips, $G$ and $G'$ can be transformed into each other using $\mathbb{B}^2$. ■

![Figure 3.16: A diagonal flip and $\mathbb{B}^2$](image)

Note that in the transition diagram $\mathcal{D}_N$, $\mathcal{D}_{3n-6,n}$ and $\mathcal{D}_{m,3m-6}$ are the sets of isolated vertices under the three transformations $A$, $B$ and $C$, but by Proposition 3.11, we can make the two diagrams to be connected by using $\mathbb{B}^2$.

### 3.3.2 Hexangulations in $\mathcal{H}_{m,n}$ with $m \neq 3n-6$ (or $n \neq 3m-6$)

In this subsection, we prove (3-2). Moreover, throughout this subsection, we only deal with $m \neq 3n-6$ by symmetry. A $C$-path is a path of length three such that the two middle vertices have degree two, and that the two ends have degree at least three. It is easy to see that $C$ can be always applied to $C$-paths since $C$ preserves the simpleness and the 2-connectedness (otherwise, after apply $C$ shown in Figure 2.15, we have a 3-cycle, which contradicts to the bipartiteness property).

Lemma 3.12 Let $G$ be a hexangulation in $\mathcal{H}_{m,n}$ with $m \neq 3n-6$. If $G$ is not the standard form, then there exists a face $f$ of $G$ with $\partial f = xaybzc$ such that $\deg(x) \geq 3$, $\deg(a) \geq 3$ and $\deg(y) = 2$.

Proof. Let $B_3$ and $W_3$ be the set of black vertices of degree at least 3 and that of white vertices, respectively. Then, by Proposition 3.10, $G$ is not a 1-subdivided triangulation, and hence we have $B_3 \neq \emptyset$ and $W_3 \neq \emptyset$. 63
Firstly we show that there exists an edge $xa$ such that $x \in B_3$ and $a \in W_3$. Let $P$ be a shortest path which connects two vertices $x \in B_3$ and $a \in W_3$. Note that the length of $P$ is one or three since $G$ is a hexangulation. Moreover, $G$ is the standard form if the length of $P$ is three. Hence, $G$ has an edge $xa$ such that $x \in B_3$ and $a \in W_3$ by the assumption.

Secondly we show that there exists a face $f$ required in the lemma. Suppose that there exists no such face in $G$. Let $f$ be a face, where $\partial f = xaybzc$ with $x, y, z \in B$ and $a, b, c \in W$ such that $x \in B_3$ and $a \in W_3$. Observe that if at least one of $y, b, z, c$ has degree 2, then $f$ must be a required face. Hence we may suppose that each vertex of $\partial f$ has degree at least three. Now we focus on a face $f'$ neighboring to $f$ and sharing an edge with $\partial f$, and consider whether $f'$ is a required face. By repeating the above argument, we can find a vertex of degree 2 since $2|E(G)| = 3|V(G)| - 6 < 3|V(G)|$.

The following two lemmas are important to prove (3-2). Let $G$ be a hexangulation with a C-path $P$ and let $f$ be a face of $G$. We define that $P$ is moved to $f$ if $G$ can be transformed into a hexangulation obtained from $G$ by removing $P$ and adding a C-path to the interior of $f$ by a repeated application of transformations. Note that each arrow represents $A$ in Figures 3.17 to 3.32. Moreover, we omit to describe “preserving the simpleness and 2-connectedness” in those lemmas and their proofs because it is easily checked by the figures.

**Lemma 3.13** Let $G$ be a hexangulation. For a hexangulation obtained from $G$ by adding a C-path $P$ to a face of $G$, $P$ can be moved to any other face of $G$ by $A$ and $C$.

**Proof.** Let $\Gamma = e_1e_2e_3e_4e_5e_6$ be the hexagonal region consisting of two hexagonal faces sharing $P$, where $e_i$ is an edge of $G$ for each $i$. Let $f$ be a neighboring face of $\Gamma$ and without loss of generality, we may suppose $e_1 \in f \cap \Gamma$ and $e_6 \notin f \cap \Gamma$ (otherwise, that is, if $e_k \in f \cap \Gamma$ for each $k$, then the lemma holds since $G$ is the standard form with 8 vertices). To prove the lemma, it is enough to consider the following four cases. (There are three kinds of the position of $P$ in $\Gamma$ (or $f$). However, we can choose any one among them by applied $C$ to $P$ in this proof.)

![Figure 3.17: Case 1](image1.png) ![Figure 3.18: Case 2 (i)](image2.png)

**Case 1.** $e_k \in f \cap \Gamma$ for $k = 1, 2, 3, 4$

By the simpleness property, we have $e_5 \notin f \cap \Gamma$. In this case, we can move $P$ to $f$ by modifying the contour of $\Gamma$ as shown in Figure 3.17.

**Case 2.** $e_k \notin f \cap \Gamma$ for $k = 1, 2, 3$ and $e_4 \notin f \cap \Gamma$
By the simpleness, we have $e_5 \notin f \cap \Gamma$. Two cases arise by symmetry. In the first case, we can move $P$ to $f$ similarly to Case 1 (see Figure 3.18). In the other case, we can move $P$ to $f$ only by $A$ as shown in Figure 3.19.

![Figure 3.19: Case 2 (ii)](image1)

**Case 3.** $e_1, e_2 \in f \cap \Gamma$ and $e_3 \notin f \cap \Gamma$

We can move $P$ to $f$ only by $A$ as shown in Figure 3.20. (Note that this operation is applicable even if $e_4$ or $e_5 \in f \cap \Gamma$. Moreover, the operation preserves the simpleness and 2-connectedness since the intermediate vertices of $P$ and the neighbors of $v$ which are not contained in $\Gamma$ are separated by a cycle. For other cases (in Lemmas 3.13 and 3.14), we similarly preserve the properties by Jordan Curve Theorem.)

![Figure 3.20: Case 3 (The dots arrow means repeating $A$.)](image2)

**Case 4.** $e_1 \in f \cap \Gamma$ and $e_2 \notin f \cap \Gamma$

Figure 3.21 shows a sequence of transformations which carries $P$ from $\Gamma$ to $f$. Depending on whether $a = x$ or not, we apply distinct operations. By planarity, we note that $by \notin E(G)$ when $a = x$. (Note that this operation is applicable even if $e_3, e_4$ or $e_5 \in f \cap \Gamma$.)

Therefore, by repeating the above operations, a C-path can be moved to any other face only by $A$ and $C$. ■

We denote by $SP$ a hexangulation obtained from the standard form by a single application of $C$ as shown in the right hand of Figure 3.22. We note that $SP$ cannot be transformed into the standard form only by $A$ since the standard form admits only $C$ by (1-1).

**Lemma 3.14** Let $G$ be a hexangulation. Suppose that $G$ has a C-path $P = abcd$, and let $\Gamma$ denote the hexagonal region consisting of the two hexagonal faces sharing $P$, where $\partial \Gamma = av_1v_2v_3v_4$. If $\deg(v_i) \geq 3$ for some $i \in \{1, 2, 3, 4\}$, then we can rotate $P$ to both directions by a sequence of $A$, unless $G$ is isomorphic to $SP$ (shown in Figure 3.22).
Proof. For convenience, we rename the vertices, as follows. Let $P = yuvb_2$ and $\partial \Gamma = xb_1b_2b_3b_4y$ and $\deg(x) \geq 3$ by symmetry (then, it is clear that $G$ is not the standard form). Let $f$ be a face sharing $xy$ with $\Gamma$, where $\partial f = xa_i a_2 a_3 a_4 y$. Moreover, for applying $\mathbb{C}$ to $P$, the clockwise rotation is called an $\alpha$-rotation and the other is called a $\beta$-rotation. Two regions $R$ and $R'$ are said to be 1-adjacent if $R$ shares only one edge with $R'$ and they do not share any vertex except the endpoints of the shared edge.

Claim 1. If $f$ and $\Gamma$ are 1-adjacent, then we can rotate $P$ to both directions only by $\mathbb{A}$.

Proof of Claim 1. Since $f$ and $\Gamma$ are 1-adjacent, we have $a_i \neq b_j$ for each $i, j = 1, 2, 3, 4$. As shown in Figure 3.23, we can rotate $P$ to both directions only by $\mathbb{A}$. In the form $K$, if $a_1$ and $b_4$ are adjacent, then we can apply $\mathbb{A}$ clockwise three times as shown in Figure 3.24 since a 4-cycle $a_1 a_2 v b_4$ separates $x, u, b_1$ and $y, a_3, a_4$ by Jordan Curve Theorem. (Moreover, it is easy to check that we can rotate $P$ to both directions only by $\mathbb{A}$ even if the two ends of $P$ are $b_1$ and $b_4$.) $\diamond$

Hence we may suppose that $f$ and $\Gamma$ are not 1-adjacent. Moreover, since we can rotate $P$ to both directions only by $\mathbb{A}$ if the operations in Claim 1 are applicable, we suppose that they are not applicable. Then, since two vertices with distinct colors do not coincide,
we might have $a_2 = b_2$ or $a_2 = b_4$ or $a_4 = b_2$ or $a_1 = b_3$ or $a_4 = b_4$ (that is, $\deg(y) = 3$), where in Figure 3.23:

if $a_2 = b_2$ then the operation (2) breaks the simpleness,

if $a_2 = b_4$ then the operation (4) breaks the simpleness,

if $a_4 = b_2$ then the operation (8) breaks the simpleness,

if $a_1 = b_3$ then the operation (9) breaks the simpleness and,

if $a_4 = b_4$ (that is, $\deg(y) = 3$) then the operations (4) and (8) break the 2-connectedness.

However, it is easy to see that the case $a_1 = b_3$ is the same as the form $X$ in Figure 3.27 and in the case $a_2 = b_4$, there exists a face $f'$ in the interior of the region $ya_4a_3b_4$ such that $f'$ and $\Gamma$ are 1-adjacent. Therefore, it suffices to consider the following cases.

Figure 3.23: Claim 1

Case 1. $a_2 = b_2$ (Figure 3.25)
In this case, as in Figure 3.25, we can rotate $P$ to both directions only by $A$. In Figure 3.25, we first exchange $xy$ with $b_1a_4$ by $A$. Then, the new hexagon $a_2b_1a_4yb_3b_2$, say $\Gamma'$, and $f$ are 1-adjacent. Hence we can apply Claim 1 to $\Gamma'$ and $f$.

**Case 2:** $a_4 = b_2$ (Figures 3.26 and 3.27)

Let $f' = b_1b_2p_1p_2p_3p_4$ be a face sharing $b_1b_2$ with $\Gamma$. If $\deg(b_1) \geq 3$, then it is not difficult to check that we can apply the operations in Claim 1 even if $x = p_2$ or $x = p_4$. Thus we may suppose $\deg(b_1) = 2$. Then we can rotate $P$ to both directions only by $A$ as shown in Figures 3.26 and 3.27. Note that we now have $a_3 \neq b_1$ since $\deg(b_1) = 2$.

Finally, we consider the case $a_4 = b_1$. However, since we do not have enough informations, we separate the cases depending $\deg(b_1)$, $\deg(b_2)$ and $\deg(b_3)$.
Case 3: $a_4 = b_4$, $\deg(b_1) = \deg(b_3) = 2$ and $\deg(b_2) = 3$ (Figure 3.28)

In this case, we can rotate $P$ to both directions only by $A$ as shown in Figure 3.28.
Case 4: $a_4 = b_4$, $\deg(b_3) = 2$, $\deg(b_2) = 3$ and $\deg(b_1) \geq 3$ (Figure 3.29)

By symmetry, this case is the same as case $a_4 = b_4$, $\deg(b_1) = 2$, $\deg(b_2) = 3$ and $\deg(b_3) \geq 3$. In this case, we can obtain the $\beta$-rotation similarly to Case 3 and the $\alpha$-rotation shown in Figure 3.29 by using the hexagonal face sharing $b_1x$ with $\Gamma$. If $q_2 = b_4$, then there exists a face $f'' = b_1xq_1q_2r_1r_2$ sharing $b_1x$ with $\Gamma$ such that $r_2 \neq b_4$. In this case, after applying $A$ to $b_1x$ to join $r_2$ and $y$, we apply Claim 1.

Case 5: $a_4 = b_4$, $\deg(b_2) = 3$, $\deg(b_1) \geq 3$ and $\deg(b_3) \geq 3$ (Figure 3.30)

In this case, two cases arise. In both cases, after applying $A$ once, we apply Claim 1 (see Figure 3.30, the operation is applicable even if $r = b_3$).
Figure 3.30: Case 5

**Case 6:** $a_4 = b_4$, $\text{deg}(b_2) \geq 4$, $\text{deg}(b_3) \geq 3$ and $\text{deg}(b_1) = 2$ (Figures 3.31 and 3.32)

By symmetry, this case is the same as case $a_4 = b_4$, $\text{deg}(b_2) \geq 4$, $\text{deg}(b_1) \geq 3$ and $\text{deg}(b_3) = 2$. Similarly to the argument in Case 2, we may suppose $\text{deg}(b_1) = 2$. If the operation in Figure 3.31 is applicable, then we can rotate $P$ to both directions only by $\alpha$ since we can apply Claim 1. Otherwise, as shown in Figure 3.32, we can obtain the $\alpha$-rotation and apply Claim 1. (Note that we now have $\text{deg}(b_1) = 2$ similarly to Case 2.)

**Figure 3.31: Case 6 (i)**

Finally, we shall describe the case $\text{deg}(y) = 3$ ($a_4 = b_4$), $\text{deg}(b_4) = 2$ ($a_3 = b_3$), $\text{deg}(b_2) = 3$ and $\text{deg}(b_1) = 2$. In this case, we cannot rotate $P$ only by $\alpha$ around $\Gamma$ such that the end points of $P$ are $x$ and $b_3$. Thus we consider the neighbor face $f'$ sharing a path $xyb_4b_3$ with $\Gamma$, where $\partial f' = xu_1u_2b_3y$. By Lemma 3.13, we can move $P$ to $f'$ only by $\alpha$ (in this case, the end points of $P$ are $y$ and $u_2$ by Figure 3.19). Then we regard $f'$ as $\Gamma$ and consider whether $P$ can be rotated by $\alpha$ such that the end points of $P$ are $x$ and $b_3$. If it is not applicable, then we have a face $f''$ sharing a path $xu_1u_2b_3$ with $f'$. Hence we move $P$ to $f''$ by $\alpha$ by Lemma 3.13 and consider whether $P$ can be rotated by
A such that the end points of $P$ are $x$ and $b_3$. By repeating the above argument, since $G$ is neither the standard form nor $SP$, we can rotate $P$ only by $A$ such that the end points of $P$ are $x$ and $b_3$.

Therefore, the lemma holds since we can obtain the operation $C$ in both directions only by $A$ in any case. $\blacksquare$

Finally, to prove (3-2) in Theorem 3.9 by the induction on the number of vertices, we shall show the following lemma.

**Lemma 3.15** Let $G$ be a hexangulation in $H_{m,n}$ with $m \neq 3n - 6$. Then we can transform $G$ into a hexangulation with at least one $C$-path only by $A$.

**Proof.** Since the lemma holds if $G$ has a C-path, we may suppose that $G$ has no C-paths (hence, $G$ is not the standard form now). By Lemma 3.12, there exists a face $f$ with $\partial f = xaybzc$ in $G$ such that $\deg(x) \geq 3$, $\deg(a) \geq 3$ and $\deg(y) = 2$. Without loss of generality, we may suppose $x, y, z \in B$ and $a, b, c \in W$. Since $G$ has no C-path, there are four kinds of such faces as shown in Figure 3.33 up to symmetry, where “2” and “$\geq 3$” mean the degree of the corresponding vertices.

![Figure 3.33: Four kinds of $f$](image)

For the leftmost case in Figure 3.33, we can immediately obtain a C-path $P = czby$ by applying $A$ to $xa$ to make $\deg(y) = 3$. Thus we may suppose that $G$ has no such faces. We now have a face $f$ with $\partial f = xaybzc$ such that $\deg(x) \geq 3$, $\deg(a) \geq 3$, $\deg(y) = 2$ and $\deg(b) \geq 3$.

We make a C-path by reducing $\deg(a)$. We first suppose $\deg(a) = 3$. Then we can obtain a C-path $byau$ by applying $A$ to $xa$ to join $c$ and $u$, where $u$ is the neighbor of $a$ other than $x$ and $y$. If this operation is not applicable, $G$ has an edge $cu$. (See the left hand of Figure 3.34).

![Figure 3.34: The two cases of $\deg(a) = 3$](image)
Let \( f' \) be a face sharing \( xa \) with \( f \), where \( \partial f' = xauv_3v_2v_1 \). We now have \( c \neq v_1 \) by \( \deg(x) \geq 3 \) and \( y \neq v_2 \) and \( b \neq v_1, v_3 \) since a 4-cycle \( cxau \) separates \( v_1, v_2, v_3 \) and \( y, b \) in the interior and exterior by Jordan Curve Theorem. We first exchange \( xa \) with \( v_1y \) by \( \mathbb{A} \). If \( \deg(u) = 2 \), then we obtain a C-path \( yauv \) since we now have \( \deg(c) \geq 3 \) (see the left hand of Figure 3.34). Otherwise, that is, if \( \deg(u) \geq 3 \), then we can obtain a C-path \( byau \) by exchanging \( v_1y \) with \( v_2b \) by \( \mathbb{A} \) (see the right hand of Figure 3.34). It is easy to see that these operations preserve the simpleness and 2-connectedness of \( G \).

Hence we may suppose that \( \deg(a) > 3 \). Let \( F_1 \) be a face sharing \( xa \) with \( f \), where \( \partial F_1 = avq_1q_2aqx \), and let \( F_2 \) be a face sharing \( av \) with \( F_1 \), where \( \partial F_2 = avp_1p_2p_3p_4 \) (see Figure 3.35). If \( \deg(v) \geq 3 \), we switch \( av \) to \( xp_1 \) or \( p_4q_1 \) by \( \mathbb{A} \) to reduce \( \deg(a) \). At least one of them can be applicable by planarity and preserves \( \deg(y) = 2 \) since \( p_4 \neq y \).

![Figure 3.35: \( \deg(v) \geq 3 \)](image1)

![Figure 3.36: \( \deg(v) = 2 \)](image2)

Thus we may suppose \( \deg(v) = 2 \) as shown in Figure 3.36. We now have \( \deg(t) \geq 3 \) (otherwise, \( G \) has a C-path or a face dealt in the leftmost case in Figure 3.33). In this case, we apply \( \mathbb{A} \) to \( xa \) to join \( c \) and \( v \) keeping \( \deg(y) = 2 \). Then, by replacing \( x \) as \( v \), we can decrease \( \deg(a) \) keeping \( \deg(x) \geq 3 \), \( \deg(y) = 2 \) and \( \deg(b) \geq 3 \). If the operation is not applicable, then we have \( c = t \).

In this case, as in Figure 3.37, we can reduce \( \deg(a) \) only by \( \mathbb{A} \). After applying this operation, we can decrease \( \deg(a) \) keeping \( \deg(x) \geq 3 \), \( \deg(y) = 2 \) and \( \deg(b) \geq 3 \) by replacing \( v, r \) and \( q_2 \) as \( x, c \) and \( z \), respectively. Therefore, since we can decrease \( \deg(a) \) preserving \( \deg(x) \geq 3 \), \( \deg(y) = 2 \) and \( \deg(b) \geq 3 \) in all cases, we can obtain a C-path only by \( \mathbb{A} \).

We have prepared to prove the statement (3-2) in Theorem 3.9.

**Proof of (3-2).** Let \( G \) and \( G' \) be hexangulations in \( \mathcal{H}_{m,n} \) with \( m \neq 3n - 6 \) and let \( V(G) = B \cup W \) be the bipartition of \( G \). Moreover, since it is easy to see that \( G \) and \( G' \) are isomorphic if \( |V(G)| \leq 8 \), we may suppose \( |V(G)| \geq 10 \). Then, by induction on \( |V(G)| \), we shall prove that \( G \) and \( G' \) can be transformed into each other only by \( \mathbb{A} \) if neither \( G \) nor \( G' \) is the standard form.

By Lemma 3.15, since \( G \) (or \( G' \)) can be transformed into a hexangulation with at least one C-path only by \( \mathbb{A} \), we may suppose that \( G \) and \( G' \) have at least one C-path. Now, let \( P = abcd \) (resp., \( P' = ab'd'c'd' \)) be a C-path in \( G \) (resp., \( G' \)), and let \( G \setminus P \) (or \( G' \setminus P' \)) denote the hexangulation obtained from \( G \) (or \( G' \)) by removing two vertices \( b \) and \( c \) (or \( b' \) and \( c' \)). If \( |V_B(G \setminus P)| < 3|V_B(G' \setminus P)| - 6 \) holds, then \( G' \setminus P \) and \( G \setminus P' \) can be transformed into each other only by \( \mathbb{A} \) by the inductive hypothesis. Moreover, since \( P \) and \( P' \) can be
moved to everywhere and rotated to both directions only by $\mathbb{A}$ by Lemmas 3.13 and 3.14, our result follows. If $G\setminus P$ is the standard form with $|V(G\setminus P)| > 8$, then we can choose another C-path $P''$ such that $G', P''$ is not the standard form since $G$ is not the standard form and $|V(G)| \geq 10$. Hence we may suppose $|V_B(G\setminus P)| = 3|V_W(G\setminus P)| - 6$, that is, $G\setminus P$ is a 1-subdivided triangulation by Proposition 3.10. By Proposition 3.11, $G\setminus P$ and $G', P''$ can be transformed into each other only by $\mathbb{B}^2$. Moreover, we can see that $\mathbb{B}^2$ in $G\setminus P$ corresponds to applying $\mathbb{A}$ twice in $G$ by using a C-path $P$ as shown in Figure 3.38. Since we can move a C-path to everywhere by Lemma 3.13, $G$ and $G'$ can be transformed into each other only by $\mathbb{A}$. Therefore, the statement holds. 

Figure 3.38: Applying $\mathbb{B}^2$ once and $\mathbb{A}$ twice
3.3.3 Problems on diagonal transformations in hexangulations

In the end of this section, we propose open problems on diagonal transformations in hexangulations. For hexangulations on the sphere, we can make the interesting transition diagram in this chapter. In the diagram, a transformation \( C \) is needed only once to transform a hexangulation on the sphere into another, and the standard form guarantees this fact. However, for any other closed surface \( F^2 \neq S_0 \), we have not yet found a hexangulation on \( F^2 \) which requires \( C \). Therefore, there is a possibility that we do not need \( C \) for hexangulations on closed surfaces except the sphere as follows. (For non-bipartite hexangulations, we guess that we need the argument of a “cycle parity”, which is introduced in quadrangulations.)

**Conjecture 3.16** For any closed surface \( F^2 \) except the sphere, any two bipartite hexangulations on \( F^2 \) with the same and sufficiently large number of vertices can be transformed into each other only by \( A \) and \( B \).

By Theorem 4.4 (which will be introduced in the next chapter) and the relation between a diagonal flip and the transformation \( B^2 \), we can immediately prove that any two 1-subdivided triangulations on the sphere with the same bipartition size \( (B,W) \) and \( B \geq W \geq 3 \) can be transformed into each other by at most \( 6W - 30 \) \( B^2 \)'s. That is, in the diagram of hexangulations on the sphere (see Figure 3.13), the diameter of the subgraph \( H_{3W-6,W} \) (resp., \( H_{B,3B-6} \)) is at most \( 6W - 30 \) (resp., \( 6B - 30 \)). Then, we conjecture that the diameter of every other subgraph of the diagram is also \( O(B + W) \). (A function \( f(n) \) is \( O(n) \) if \( \lim_{n \to \infty} \frac{f(n)}{n} < \infty \), that is, the value is finite.)

**Conjecture 3.17** Let \( G \) and \( G' \) be hexangulations on the sphere with the same bipartition size \( (B,W) \), where \( B \geq W \) and \( B \neq 3W - 6 \). Then \( G \) and \( G' \) can be transformed into each other by \( O(B + W) \) \( A \)'s and at most one \( C \).

3.4 Note on \( N \)-angulations for \( N \geq 7 \)

By Theorem 2.22 and the results in the above sections, transition diagrams for \( N \)-angulations on the sphere are connected for any integer \( N \geq 3 \). However, for \( N \)-angulations with \( N \geq 7 \), the details of the structure of transition diagrams are not known. Then, in this section, we introduce easy observations for the transition diagram of 7-angulations on the sphere.

By the definition of diagonal transformations, there are three kinds of transformations in 7-angulations \( A_s, B_s \) and \( C_s \) shown in Figure 3.39.

![Figure 3.39: Diagonal transformations in 7-angulations](image)
For each of $A_s, B_s$ and $C_s$, there exists 7-angulation on the sphere which requires the transformation shown in Figure 3.40. Moreover, we can construct 7-angulation on the sphere which requires $B_s$ or $C_s$ as follows: Let $Q$ be a quadrangulation on the sphere with the minimum degree at least 3, and let each edge in $Q$ be colored by red. Next, by adding a blue edge to each face in $Q$, we have a triangulation $T$ on the sphere with red and blue edges. Finally, we can obtain a 7-angulation $H$ on the sphere by replacing each red (resp., blue) edge in $T$ with a path of length 2 (resp., 3), where each middle vertex has degree 2. It is easy to see that $H$ admits only transformations $B_s$ and $C_s$.

![Figure 3.40: 7-angulations requiring $A_s, B_s$ and $C_s$](image)

Therefore, we guess that the transition diagram of 7-angulations on the sphere has a similar substructure to that of pentangulations on the sphere, however, the structure will be slightly different from that of pentangulations on the sphere. Then, for the structures of transition diagrams for $N$-angulations, we would like to expect further researches on diagonal transformations in $N$-angulations on surfaces.
Chapter 4
Diameter of transition diagrams

In this chapter, we consider the diameter of transition diagrams for $N$-angulations. Let $D_n$ be a transition diagram for $N$-angulations with $n$ vertices. A distance and a diameter are defined as same as those of graphs, respectively (cf. Chapter 1), that is, a diameter of $D$ denoted by $\text{diam}(D)$ means that the maximum number of the number of diagonal transformations needed to transform an $N$-angulation into another. Then, we have a natural question: How many diagonal transformations do we need to transform two given $N$-angulations into each other? This question is closely related to an algorithm which generates a sequence of diagonal transformations from one to the other. For this problem, there are only results on triangulations and quadrangulations, and we show our result on quadrangulations on the sphere. In the final section, we propose problems on the number of diagonal transformations in $N$-angulations.

4.1 Known results on triangulations

In this section, we introduce known results on the number of diagonal flips in triangulations. Wagner’s proof [78] (cf. Page 26) suggested one of algorithms which transform a given triangulation on the sphere into the standard form $\Delta_n$ of planar triangulations with $n$ vertices shown in Figure 2.3 (note that deg($y$) = deg($z$) = $n + 2$). Moreover, it is easy to check that the algorithm will generate $O(n^2)$ diagonal flips for a given triangulation on the sphere with $n$ vertices. (Recall that a function $f(n)$ is $O(n)$ if $\lim_{n \to \infty} \frac{f(n)}{n} < \infty$, that is, the value is finite.)

Komuro [38] improved the Wagner’s algorithm and proved that his algorithm requires at most $8n - 48$ (resp., $8n - 54$) diagonal flips if $n \geq 7$ (resp., $n \geq 13$) to transform two given triangulations on the sphere into each other, where $n$ is the number of vertices. Moreover, it was proved that there exist two special triangulations on the sphere with $n$ vertices which require at least $2n - 15$ diagonal flips to transform into each other [38]. Hence, the order $O(n)$ in Komuro’s result is best possible. Afterward, Mori et al. [50] improved Komuro’s result by using a clever idea, and then, we introduce the outline of the proof in [50].

A maximal outerplane graph with $n$ vertices is a graph on the plane which consists of an outer cycle $C$ with the length $n$ (that is, there is no vertex in the interior of $C$) and
every inner face is triangular (see Figure 4.1).

![Figure 4.1: A maximal outerplane graph](image)

To improve Komuro’s result, Mori et al. focused on 4-connected triangulations on the sphere. It is well known that any 4-connected triangulation has a Hamilton cycle [76]. Hence, we can decompose a 4-connected triangulation $G$ on the sphere into two maximal outerplane graphs $G_1$ and $G_2$, which have the same Hamilton cycle. Since maximal outerplane graphs can be treated easier than triangulations on the sphere, Mori et al. first proved the following for 4-connected triangulations.

**Theorem 4.1 (Theorem 3 in [50])** Any two 4-connected triangulations $G$ and $G'$ on the sphere with $|V(G)| = |V(G')| = n \geq 6$ can be transformed into each other by at most $4n - 22$ diagonal flips.

**Sketch of the proof.** Let $G$ and $G'$ be 4-connected triangulations on the sphere with $n \geq 6$ vertices. Since $G$ has a Hamilton cycle $C$, $G$ can be decomposed into two maximal outerplane graphs $G_1$ and $G_2$, where $G_i$ has the outer cycle $C$ for each $i$ (that is, $|V(G_1)| + |V(G_2)| = 2n$). Then, Mori et al. show that we can transform $G_i$ into the canonical form of maximal outerplane graphs shown in the right hand of Figure 4.2 by at most $n - 3$ diagonal flips (note that $n - 3$ is the number of diagonals in a maximal outerplane graph).

![Figure 4.2: Transforming a maximal outerplane graph into the canonical form](image)

Note that the canonical form has a unique vertex of degree $n - 1$, which is called an apex. Therefore, for each $i$ and an edge $uv \in E(C)$, we can transform $G_1$ and $G_2$ into the canonical forms which have $u$ and $v$ as apexes, respectively. Since it is clear that the
resulting graph is the standard form of triangulations on the sphere by gluing $G_1$ and $G_2$ along $C$, we can transform any triangulation $G$ on the sphere into the standard form by at most $2n - 6$ diagonal flips. (In this case, we apply at most $4n - 12$ diagonal flips to transform $G$ into $G'$, but Mori et al. prove that the number is at most $4n - 22$ in fact by a more careful observation.)

The followings are two critical lemmas to prove the main theorem in [50]. A *separating* 3-cycle $C$ of a graph $G$ on the sphere is a 3-cycle such that $G - C$ is disconnected. Moreover, Lemma 4.3 implies that each separate 3-cycle in a triangulation can be broken by applying a diagonal flip at most once. In this thesis, we only introduce the proof of Lemma 4.3 (since the proof of Lemma 4.2 is not difficult, which uses the induction of the number of vertices).

**Lemma 4.2 (Lemma 10 in [50])** Any triangulation on the sphere with $n$ vertices has at most $n - 4$ separating 3-cycles.

**Lemma 4.3 (Lemma 11 in [50])** Any triangulation on the sphere with $n \geq 6$ vertices can be transformed into a 4-connected triangulation on the sphere by at most $n - 4$ diagonal flips.

*Proof of Lemma 4.3.* It is easy to see that every triangulation on the sphere is 3-connected, and if $\{x, y, z\}$ is a set of vertices such that $G - \{x, y, z\}$ is disconnected, then $x, y$ and $z$ are contained in the same 3-cycle. Since $G$ has at most $n - 4$ separating 3-cycles by Lemma 4.2, it suffices to prove that $G$ has an edge $e$ such that the diagonal flip of $e$ decreases the number of separating 3-cycles by at least one.

Let $C = xyz$ be a separating 3-cycle in $G$ and let $e = xy$. In this case, we may suppose that if $G$ has an edge shared by at least two separating 3-cycles, then we choose such an edge as $e$. Let $xayb$ be the quadrilateral formed by two triangular faces sharing $e$, where $a$ and $b$ are in the interior and the exterior of $C$, respectively. Moreover, let $G'$ be the resulting graph obtained from $G$ by the diagonal flip of $e$ replacing $xy$ and $ab$.

Now, we show that no new separating 3-cycle has arisen in $G'$. Suppose that $G'$ has a new separating 3-cycle $C'$, and then, $C'$ now contains both $a$ and $b$ since $G'$ is also 3-connected. Thus, we put $C' = abc$. Since $a$ was contained in a component of $G - \{x, y, z\}$, we must have $z = c$. Since $|V(G)| \geq 6$, either $xza, yza, xzb$ or $yzb$ is a separating 3-cycle. In these cases, $xz$ or $zy$ is an edge included in at least two separating 3-cycles, but $xy$ is contained in only one separating 3-cycle, which contradicts to the choice of $e$. Therefore, no new separating 3-cycle has arisen in $G'$.

Then, Mori et al. finally proved the following.

**Theorem 4.4 (Mori et al. [50])** Any two triangulations $G$ and $G'$ on the sphere with $|V(G)| = |V(G')| = n \geq 6$ can be transformed into each other by at most $6n - 30$ diagonal flips.

*Outline of the proof.* Let $G$ and $H$ be two triangulations on the sphere with $n \geq 6$ vertices. By Lemma 4.3, $G$ and $H$ can be transformed into 4-connected triangulations $G'$ and $H'$.
on the sphere by at most \( n - 4 \) diagonal flips, respectively. Then, by Theorem 4.1, \( G' \) and \( H' \) can be transformed into each other by at most \( 4n - 22 \) diagonal flips. Hence, we can transform \( G \) into \( H \) by at most \((n - 4) + (n - 4) + (4n - 22) = 6n - 30 \) diagonal flips. ■

Moreover, for triangulations on other surfaces, the diameter of transition diagrams is estimated as follows. (The estimation in Theorem 4.7 is not exactly determined since the paper is preprint.)

***Theorem 4.5 (Mori and Nakamoto [48])** Any two triangulations \( G \) and \( G' \) on the projective plane with \( |V(G)| = |V(G')| = n \) can be transformed into each other by at most \( 8n^2 - 26 \) diagonal flips.

***Theorem 4.6 (Sekine [73])** Any two triangulations \( G \) and \( G' \) on the torus with \( |V(G)| = |V(G')| = n \) can be transformed into each other by at most \( \frac{70}{3}n - 122 \) diagonal flips.

***Theorem 4.7 (Mori and Nakamoto [49])** For any closed surface \( F^2 \neq S_0, N_1 \), there exists a positive integer \( N(F^2) \) such that any two triangulations \( G \) and \( G' \) on \( F^2 \) with \( |V(G)| = |V(G')| = n \geq N(F^2) \) can be transformed into each other by \( O(n) \) diagonal flips.

Recall that \( T_{n,F^2} \) is the transition diagram of triangulations on a closed surface \( F^2 \) with \( n \) vertices. By Theorem 4.4 and Komuro’s result, we have \( 2n - 15 \leq \text{diam}(T_{n,S_0}) \leq 6n - 30 \). Moreover, by Theorems 4.5, 4.6 and 4.7, we have \( \text{diam}(T_{n,N_1}) \leq 8n - 26 \), \( \text{diam}(T_{n,S_1}) \leq \frac{70}{3}n - 122 \) and \( \text{diam}(T_{n,F^2}) = O(n) \), respectively. (Note that each \( n \) is sufficiently large.)

### 4.2 Results on quadrangulations

In this and next sections, a quadrangulation on the sphere is simply called a quadrangulation since we only deal with that on the sphere. Similarly to triangulations, there is a result on the number of diagonal transformations to transform a given quadrangulation into another as follows.

***Theorem 4.8 (Nakamoto and Suzuki [58])** Any two quadrangulations \( G \) and \( G' \) with \( n = |V(G)| = |V(G')| \geq 6 \) can be transformed into each other by at most \( 6n - 32 \) diagonal transformations.

In the proof of Theorem 4.8, Nakamoto and Suzuki [58] proved that any quadrangulation with \( n \) vertices can be transformed into the standard form \( S_{2,n-2} \) shown in Figure 4.3. Therefore, for every two integers \( B \neq 2, n - 2 \) and \( W \neq 2, n - 2 \), if a quadrangulation \( Q \) with the bipartition size \((B,W)\) is given, then we have to apply a diagonal rotation to \( Q \) for transforming \( Q \) into \( S_{2,n-2} \). In other words, even if two given quadrangulations \( G \) and \( G' \) have the same bipartition size, we change the bipartition size of \( G \) (or \( G' \)) by a diagonal rotation to transform \( G \) into \( G' \) by \( O(n) \) diagonal transformations. However, in fact, if \( G \) and \( G' \) have the same bipartition size, we do not need a diagonal rotation by Theorem 2.18. Moreover, the number of diagonal slides in the proof of Theorem 2.18 is

80
$O(BW)$, where $B$ and $W$ are the number of black vertices and white ones, respectively. Then, we re-prove Theorem 2.18 for the spherical case by $O(B + W)$ diagonal slides as follows.

**Theorem 4.9** Any two quadrangulations $G$ and $G'$ with $|V_B(G)| = |V_B(G')| \geq |V_W(G)| = |V_W(G')| \geq 3$ can be transformed into each other by at most $10|V_B(G)| + 16|V_W(G)| - 64$ diagonal slides.

By this theorem, we also have the following corollary.

**Corollary 4.10** Let $G$ and $G'$ be quadrangulations with $k \geq 6$ vertices and $|V_B(G)| \geq |V_B(G')|$. Then there exists a sequence of diagonal transformations of length at most $13k - 64$ from $G$ to $G'$ in which exactly $|V_B(G)| - |V_B(G')|$ diagonal rotations are applied.

Now, we construct two quadrangulations which need many diagonal transformations to transform the quadrangulation into the other, and show that the linear order of the bound in Theorem 4.9 (or Corollary 4.10) is best possible with respect to the number of vertices.

Let $G$ and $G'$ be quadrangulations with the same bipartition size such that $G \cong S_{m,n}$ (the standard form) and $G' \cong G_{m,n}$ shown in Figure 4.4, where $m = 2k + l$, $n = 2k$ and $k$ is sufficiently large. Now, by considering the maximum degree of $G$ and $G'$, we can see that $G$ has two black $n$-vertices and two white $m$-vertices and $G'$ has two black 4-vertices and two white $(l + 3)$-vertices. Let $x$ and $y$ be black 4-vertices, and let $u$ and $v$ be white $(l + 3)$-vertices in $G'$. Then, we consider to make $\text{deg}(x) = \text{deg}(y) = n$ and $\text{deg}(u) = \text{deg}(v) = m$ only by diagonal slides. Note that each diagonal slide cannot simultaneously increase the degree of two black (or white) vertices. Moreover, at most two diagonal slides can simultaneously increase the degree of two of $x, y, u$ and $v$ by the simpleness. Thus, we need at least $2(n - 4) - 2 + 2\{m - (l + 3)\} - 2 = 2m + 2n - 2l - 18$ diagonal slides to transform $G$ into $G'$. (Note that for any way of transformations, the number of transformations deforming $G$ into $G'$ is at least $2m + 2n - 2l - 18$ since we now choose vertices of the maximum degree.) Therefore, the linear order of the bound in Theorem 4.9 is best possible.

Finally, we consider the diameter of the transition diagram of quadrangulations on the sphere (for the definition of the transition diagram for quadrangulations, see Page 77).
By Theorem 4.8 and 4.9, we have diam($Q_{M,S_0}$) ≤ 6n − 32 for n ≥ 6 and diam($Q_{B,W,S_0}$) ≤ 10B + 16W − 64 for B ≥ W ≥ 3, respectively. Moreover, by Corollary 4.10, there are two paths P and P′ in $Q_{M,S_0}$ transforming an element $G ∈ Q_{M,S_0}$ into another $G' ∈ Q_{M,S_0}$, where |P| ≤ 6n − 32 and |P′| ≤ 13n − 64, and the number of edges implying diagonal rotations is at most $|V_B(G)| - |V_B(G')|$ on P′.

4.3 Proof of Theorem 4.9 and Corollary 4.10

In this section, we shall prove Theorem 4.9 and Corollary 4.10. We first prepare several lemmas to prove the theorems. Throughout the following lemmas, G denotes a quadrangulation with |B| ≥ |W|.

**Lemma 4.11** There exists a vertex $x ∈ B$ such that deg($x$) ≤ 3.

*Proof.* We suppose that deg($x$) ≥ 4 for any vertex $x ∈ B$. Observe $4|F(G)| = 2|E(G)|$, $|V(G)| = |B| + |W|$, and $|E(G)| ≥ 4|B|$ by the assumption. By Euler’s formula, we have $|E(G)| ≤ 2|V(G)| − 4$, and hence,


This inequality contradicts $|B| ≥ |W|$.

**Lemma 4.12** Let $e = xy ∈ E(G)$. If the degree of $x$ and $y$ are at least 3, then $e$ can be flipped by a diagonal slide.

*Proof.* Let $xaby$ and $xcdy$ be two faces sharing the edge $xy$ (note that $a ≠ c$ and $b ≠ d$ since deg($x$) ≥ 3 and deg($y$) ≥ 3). Consider to switch $xy$ to $ad$ by a diagonal slide. If this operation is not applicable, then $ad ∈ E(G)$. In this case, we can switch $xy$ to $cb$ by a diagonal slide since if $cb ∈ E(G)$, then $G$ contains a complete bipartite graph $K_{3,3}$ as a subgraph, and so this contradicts $G$ is planar.

**Lemma 4.13** At most one diagonal slide yields a 2-vertex in $B$. 

82
Proof. If \( G \) has a 2-vertex \( v \in B \), then we are done. Hence we may suppose that there exists \( v \in B \) with \( \deg(v) = 3 \) by Lemma 4.11. For three vertices \( w_1, w_2, w_3 \in N(v) \), if \( \deg(w_i) = 2 \) for each \( i \in \{1, 2, 3\} \), then \( G \) has exactly three faces \( vw_1bw_2, vw_2bw_3 \) and \( vw_3bw_1 \) meeting at \( v \in B \), and \( B = \{v, b\}, W = \{w_1, w_2, w_3\} \). However, this contradicts \( |B| \geq |W| \), and hence, we may suppose that \( \deg(w_1) \geq 3 \). Then, by Lemma 4.12, we can switch an edge \( vw_1 \) to reduce \( \deg(v) \) by one. Hence, the lemma holds.

The following is essential for proving Theorem 4.9.

**Proposition 4.14** Let \( G \) be a quadrangulation with \( |B| = m \geq |W| = n \geq 3 \). Then \( G \) can be transformed into the standard form by at most \( 5m + 8n - 32 \) diagonal slides.

**Proof.** By Lemma 4.13, we may suppose that there exists \( b_1 \in B \) such that \( \deg(b_1) = 2 \) after we apply at most one diagonal slide. Then, we first regard a face \( ub_1vb_m \) as an outer face as shown in Figure 4.5. Consider the constant \( d(u, v) = 2 \deg(u) + \deg(v) \). Since \( \deg(u) \geq 3 \) and \( \deg(v) \geq 3 \), we have \( d(u, v) \geq 9 \).

Let \( ub_1vb_2 \) be the face sharing the path \( ub_1v \) with the outer face \( ub_1vb_m \) and let \( w \) be a neighbor of \( b_2 \) which is next to \( v \) with respect to the anti-clockwise rotation around \( b_2 \). If \( w = u \), then we consider a next vertex \( b_3 \in B \) since we have \( \deg(b_2) = 2 \), where a face \( ub_2vb_3 \) shares a path \( ub_2v \) with the face \( ub_1vb_2 \) (in this case, we regard \( b_2 \) and \( b_3 \) as \( b_1 \) and \( b_2 \), respectively). Hence we may suppose \( w \neq u \). Let \( vb_2wx \) and \( wb_2w'y \) be faces sharing the edge \( b_2w \) as shown in Figure 4.6.

The first case is when \( \deg(w) \geq 3 \) (see Figure 4.6). Switch \( b_2w \) to \( vy \) by a diagonal slide. This operation increases \( d(u, v) \) by one and decreases \( \deg(b_2) \) by one. If this operation is not applicable, then there already exists an edge \( vy \) (note that if \( b_m = x \), then we can apply the operation). In this case, we can slide \( b_2v \) to \( ux \) by a diagonal slide by Lemma 4.12 (note that it also preserves \( \deg(v) \geq 3 \) since \( x \) does not coincide with \( b_m \) even if \( y = b_m \)), and this operation increases \( d(u, v) \) by one since \( \deg(u) \) increases by one. After applying this operation, we replace \( x \) with \( b_2 \). Thus, we can increase \( d(u, v) \) by one.

The second case is when \( \deg(w) = 2 \) (see Figure 4.7; this configuration is obtained from Figure 4.6 by identifying \( x \) and \( y \) and replacing them by \( z \)). In this case, as shown in Figure 4.8, we move \( w \) to the quadrilateral region \( ub_1vb_m \) by exactly four diagonal slides by Lemma 2.15. After this operation, \( d(u, v) \) is un-changed but \( \deg(b_2) \) is decreased by one.
Then, since we eventually have $\deg(b_2) = 2$, we continue the above operations for $b_3$, where $ub_2vb_3$ is a face sharing a path $ub_2v$ with $ub_1vb_2$. (In this case, if the above second case appears, then we move a white 2-vertex to the quadrilateral region $ub_1vb_2$ by exactly four diagonal slides. In general, in the above procedures for $b_i$, we move a white 2-vertex to the quadrilateral region $ub_{i-2}vb_{i-1}$ by exactly four diagonal slides when the second case happens.) By repeating the above operations, we can transform $G$ into a special form, say $H$, with $d(u, v) = 3m$ as shown in Figure 4.9 since $u$ and $v$ are adjacent to all black vertices.

Next, we transform $H$ into the standard form. Since $|W| \geq 3$, one of the quadrilateral regions $ub_ivb_{i+1}$, say $ub_mvb_1$, includes at least one white 2-vertices, say $w_1, \ldots, w_k$ ($1 \leq k \leq 84$).
2), in the interior, where \(vb_1w_1b_m\) and \(ub_1wb_m\) are faces of \(H\) (see Figure 4.9). Then, we can move each of white 2-vertices in two quadrilateral regions \(ub_1vb_2\) and \(ub_2vb_3\) into the quadrilateral region \(vb_1w_1b_m\) by at most four diagonal slides as shown in Figure 4.10. Hence we eventually have \(\deg(b_2) = 2\), and then we can move \(b_2\) into the quadrilateral region \(ub_1wb_m\) by exactly two diagonal slides as shown in Figure 4.11. Hence, applying the above operations repeatedly, we can transform \(G\) into the standard form \(S_{m,n}\).

Finally, we estimate the total number of diagonal slides applied. In the procedures transforming \(G\) into \(H\), since one diagonal slide increases \(d(u,v)\) by exactly one, and since \(d(u,v) \geq 9\) and \(d(u,v) = 3m\), we applied at most \(3m-9\) diagonal slides. Moreover, in the process, we moved each of the white 2-vertices except \(u\) and \(v\) by exactly four diagonal slides. In the process transforming \(H\) into the standard form, each black (resp., white) 2-vertex except \(b_1\) and \(b_m\) (resp., \(u, v\) and \(w\)) is moved to the specific quadrilateral regions \(ub_1w_1b_m\) (resp., \(vb_1wb_m\)) by at most two (resp., four) diagonal slides. Then, since we apply at most one diagonal slide to obtain a black 2-vertex by Lemma 4.13, we have

\[(3m - 9) + 4(n - 2) + 2(m - 2) + 4(n - 3) + 1 = 5m + 8n - 32.\]

**Proof of Theorem 4.9.** By Proposition 4.14, we can transform \(G\) and \(G'\) into the same standard form \(S_{m,n}\) by at most \(5m + 8n - 32\) diagonal slides. Therefore, the number of diagonal slides to transform \(G\) into \(G'\) is at most \(10m + 16n - 64\) in total.

Finally, we shall prove Corollary 4.10.
Proof of Corollary 4.10. Let \((m, n)\) and \((m', n')\) be the bipartition sizes of \(G\) and \(G'\), respectively. By Proposition 4.14, \(G\) and \(G'\) can be transformed into \(S_{m,n}\) and \(S_{m',n'}\) only by diagonal slides, respectively. As shown in Figure 4.12, we can transform \(S_{k,l}\) into \(S_{k+1,l-1}\) (or \(S_{k-1,l+1}\)) by two diagonal slides and one diagonal rotation.

![Figure 4.12: Transforming a standard form into another standard form](image)

Therefore, since we can transform \(S_{m,n}\) into \(S_{m',n'}\) by applying the operation shown in Figure 4.12 repeatedly, there exists a sequence of diagonal transformations, where the number of diagonal rotations is exactly \(|B| - |B'|\).

Finally, we estimate the length of the sequence. In this proof, we suppose that \(m \geq n\). (Since we can easily prove the case \(m < n\) similarly to the following proof method, we entrust the remaining case to the reader.) Then, we first transform \(G\) and \(G'\) into \(S_{m,n}\) and \(S_{m',n'}\) by at most \(5m + 8n - 32\) and at most \(5m' + 8n' - 32\) (or \(5n' + 8m' - 32\)) diagonal slides, respectively. Moreover, we apply \(3(m - m')\) diagonal transformations to transform \(S_{m,n}\) into \(S_{m',n'}\). Then, we have

\[
(5m + 8n - 32) + (5m' + 8n' - 32) + 3(m - m') = (8m + 8n) + (2m' + 2n') + 6n' - 64
\leq 8k + 2k + (6 \times \frac{k}{2}) - 64 = 13k - 64, \text{ or}
\]

\[
(5m + 8n - 32) + (5n' + 8m' - 32) + 3(m - m') = (8m + 8n) + (5n' + 5m') - 64
\leq 8k + 5k - 64 = 13k - 64.
\]

In both cases, since the length of the sequence is at most \(13k - 64\), we complete the proof. \(\blacksquare\)
4.4 The number of diagonal transformations in $N$-angulations for $N \geq 5$

In this section, we introduce several problems on the number of diagonal transformations in $N$-angulations for $N \geq 5$. For triangulations and quadrangulations on the sphere, there are results for the diameter of transition diagrams. However, it has not yet been known whether the estimations are best possible or not. Hence, to determine the best value of those diameters is a big problem. Moreover, we conjecture that for any closed surface $F^2$, there exists a positive integer $N(F^2)$ such that any two bipartite (or non-bipartite) quadrangulations on $F^2$ with $n \geq N(F^2)$ vertices can be transformed into each other by $O(n)$ diagonal transformations (of course, we will need the argument of a “cycle parity” in the non-bipartite case). Then, for $N$-angulations with $N \geq 5$, we daringly propose the following conjecture.

**Conjecture 4.15** For any closed surface $F^2$ and any integer $N \geq 5$, any two $N$-angulations on the sphere with $n$ vertices can be transformed into each other by $O(n)$ diagonal transformations.

In the transition diagram of pentangulations on the sphere (cf. Chapter 3), almost pentangulations can be transformed into each other only by $A$, and $B$ is needed only when we transform the standard form of pentangulations on the sphere into another. Hence, we conjecture the following. (For hexangulations on the sphere, we conjecture the similar propositions, see Page 75.)

**Conjecture 4.16** Any two pentangulations on the sphere with $n$ vertices can be transformed into each other by $O(n)$ $A$’s if they are not isomorphic to the standard form.
Chapter 5

A survey of local transformations

In this chapter, we shall introduce the research on diagonal transformations in $N$-angulations preserving specified properties and on local transformations in graphs other than diagonal transformations. We also show our results on a few of those transformations. Moreover, in several subsections, we propose open problems for those topics.

5.1 Diagonal transformations preserving specified properties

In this section, we shall introduce known results on diagonal transformations in $N$-angulations preserving specified properties. That is, we consider whether or not two given $N$-angulations with the specified property $\mathcal{P}$ can be transformed into each other by diagonal transformations preserving $\mathcal{P}$.

5.1.1 Minimum degree conditions

In this subsection, we consider the equivalence for $N$-angulations with respect to the minimum degree conditions. Let $N_d(F^2)$ be a positive integer such that any two triangulations on a closed surface $F^2$ with $N \geq N_d(F^2)$ vertices and the minimum degree at least $d$ can be transformed into each other by diagonal flips through those triangulations. By Theorem 2.7, there exists a finite constant $N_3(F^2)$ (note that $N_3(F^2)$ implies $N(F^2)$ in Theorem 2.7). Moreover, Komuro, Nakamoto and Negami proved that there exists a finite constant $N_4(F^2)$ for any closed surface $F^2$ as follows.

Theorem 5.1 (Komuro et al. [39]) For any closed surface $F^2$, there exists a positive integer $N_4(F^2)$ such that any two triangulations $G$ and $G'$ on $F^2$ can be transformed into each other preserving $\delta(G) \geq 4$ and $\delta(G') \geq 4$ if $|V(G)| = |V(G')| \geq N_4(F^2)$.

It is not difficult to see that there are only finitely many graphs embedded on a closed surface $F^2$ with the minimum degree at least 7. (It follows from Euler’s formula that $|V(G)| \leq 6|\varepsilon(F^2)|$ for such a graph $G$.) Then, how about $N_5(F^2)$ and $N_6(F^2)$?

A graph is said to be $d$-covered if each edge is incident to a vertex of degree $d$. A triangulation $G$ on a closed surface is said to be $d$-frozen if $\delta(G) \geq d$ and if any diagonal
flip in \( G \) results in a non-simple graph or decreases its minimum degree to less than \( d \). Thus, no diagonal flip is applicable to a \( d \)-frozen triangulation if we have to keep its minimum degree at least \( d \). Komuro et al. [39] presented \( d \)-frozen triangulations on any closed surface \( F^2 \) with an arbitrarily large number of vertices for \( d = 5 \) and \( d = 6 \) (note that \( \varepsilon(F^2) \leq 0 \) in the case when \( d = 6 \)). Therefore, \( N_5(F^2) \) and \( N_6(F^2) \) do not exist as finite constants for any closed surface \( F^2 \).

For quadrangulations on surfaces, let us focus on the similar problem to the above. To consider diagonal transformations in quadrangulations with the minimum degree at least 3, we first introduce an interesting graph called a radial graph. (Note that since a quadrangulation \( G \) on a closed surface \( F^2 \) has a vertex of degree at most 3 if \( \varepsilon(F^2) > 0 \) by Euler’s formula, “the minimum degree at least 3” and “the minimum degree 3” are equivalent for quadrangulations on the sphere or the projective plane.) Let \( G \) be a graph embedded on a closed surface \( F^2 \) with black vertices. Put a white vertex into each face of \( G \) and join it with black vertices of \( G \) lying along the boundary walk of the corresponding face, and delete all edges of \( G \). The resulting graph is called the radial graph of \( G \) and denoted by \( R(G) \) (which is introduced in [6]). Note that for \( G \) and its dual graph \( G^* \), \( R(G) = R(G^*) \). It is easy to see that \( R(G) \) is bipartite and each face of \( R(G) \) is quadrilateral, but \( R(G) \) is not always a quadrangulation (since \( R(G) \) is not always simple). Moreover, by the definition of the radial graph, we can easily obtain a 3-covered bipartite quadrangulation \( R(G) \) with an arbitrarily large number of vertices from a triangulation \( G \) with an arbitrarily large number of vertices.

Furthermore, we introduce the following two special 3-covered quadrangulations. In particular, one of them is a non-bipartite quadrangulation:

Embed a cycle \( v_1u_1v_2u_2\ldots v_nu_n \) \((n \geq 3)\) of even length into the sphere along the equator, put vertices \( x \) and \( y \) at the south pole and the north pole, respectively, and add edges \( xv_i \) and \( yu_i \) for \( i = 1,\ldots,n \). The resulting quadrangulation on the sphere with \( 2n+2 \) vertices is said to be the pseudo double wheel \( PW_{2n} \) (see the left hand of Figure 5.1). Next, embed a cycle \( C = v_1v_2\ldots v_{2n-1} \) \((n \geq 2)\) of odd length into the projective plane so that tubular neighborhood of \( C \) forms a Möbius band. Then, put a vertex \( x \) on the center of the unique face of the embedding and join \( x \) with \( v_i \) for all \( i \) so that the resulting graph is a quadrangulation. The resulting quadrangulation on the projective plane with \( 2n \) vertices is said to be the Möbius wheel \( MW_{2n-1} \) (see the right hand of Figure 5.1).

![Figure 5.1: PW_8 and MW_5](image-url)
By the above argument, it seems to be difficult to make theorems for diagonal transformations in quadrangulations with the minimum degree at least 3. However, the following theorem asserts that any 3-covered quadrangulation is always bipartite with two exceptions.

**Theorem 5.2 (Ando et al. [1, 2])** Let $G$ be a quadrangulation on a closed surface $F^2$ which is not isomorphic to a pseudo double wheel and a Möbius wheel. Then $G$ is 3-covered if and only if $G$ is the radial graph of a multi-triangulation on $F^2$.

By this theorem, a non-bipartite quadrangulation with minimum degree at least 3 has an edge to which a diagonal slide can be applied preserving the minimum degree if it is not a Möbius wheel. Moreover, for any closed surface $F^2$, Nakamoto [54] proved the generating theorems for quadrangulations on $F^2$ with minimum degree at least 3. Therefore, there is a possibility that for non-bipartite quadrangulations with the minimum degree at least 3, the similar theorem to Theorem 5.1 can be established, that is, we conjecture the following. (Note that we cannot use a diagonal rotation since we have to preserve the minimum degree at least 3.)

**Conjecture 5.3** For any closed surface $F^2 \neq S_0$, there exists a positive integer $M(F^2)$ such that any two non-bipartite quadrangulations $G$ and $G'$ with $N \geq M(F^2)$ vertices, the same cycle parity and the minimum degree at least 3 can be transformed into each other by diagonal slides through those quadrangulations if $G$ and $G'$ are not isomorphic to a Möbius wheel.

For pentangulations and hexangulations, we can consider the similar problems to the above. In fact, there are infinitely many pentangulations on the sphere with the minimum degree 3, but we have not found infinitely many pentangulations to which $A$ cannot be applied preserving the minimum degree at least 3. Hence, there is a problem to determine whether or not any two pentangulations on the sphere with the minimum degree 3 can be transformed into each other only by $A$ through those pentangulations. On the other hand, since any hexangulation on the sphere (or the projective plane) has a 2-vertex by Euler's formula, we can consider the similar problem only for hexangulations on closed surfaces with Euler characteristic $\varepsilon \leq 0$.

### 5.1.2 Labeled triangulations

In this subsection, we consider whether or not Theorem 2.7 can be extended to triangulations with vertex-labeling.

Let $G$ be a triangulation on a closed surface $F^2$ with $n$ vertices. An assignment $\sigma : V(G) \to \{1,2,\ldots,n\}$ is called a labeling of $G$ if $\sigma$ is a bijection between $V(G)$ and $\{1,2,\ldots,n\}$. We call a triangulation with a labeling $\sigma$ a labeled triangulation. Moreover, two labeled triangulations $G_1$ and $G_2$ are said to be strongly equivalent if $G_1$ and $G_2$ can be transformed into each other by diagonal flips so that their labelings coincide.

The following lemma is the key of the extension, and Figure 5.2 proves the lemma.

**Lemma 5.4** Let $G$ be a labeled triangulation with a vertex $v$ of degree 3. Then we can exchange the labels of $v$ and of any other vertex by diagonal flips. ■

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90
By Lemma 5.4, a vertex of degree 3 plays the role of a “label carrier”. Moreover, by Lemma 2.2, we can exchange every two labels of vertices by diagonal flips. Therefore, the following is almost a corollary of Theorem 2.7.

**Theorem 5.5 (Negami [62])** For any closed surface $F^2$, there exists a positive integer $L(F^2)$ such that two labeled triangulations $G_1$ and $G_2$ on $F^2$ are strongly equivalent if $|V(G_1)| = |V(G_2)| \geq L(F^2)$. Furthermore, we have

$$N(F^2) \leq L(F^2) \leq N(F^2) + 1.$$ 

Moreover, for several closed surfaces, Negami completely decided the value of $L(F^2)$ as follows. For more details of the arguments, see [59, 64, 65, 66].

$$L(S_0) = 4, \quad L(N_1) = 7, \quad L(S_1) = 8, \quad L(N_2) = 8 \text{ or } 9$$

Similarly to diagonal flips in usual triangulations, the number of diagonal flips in labeled triangulations is studied. For the spherical case, Gao et al. [24] estimated the number of diagonal flips in labeled triangulations as follows. However, as far as we know, there are not other results on this topic.

**Theorem 5.6 (Gao et al. [24])** Any two labeled triangulations on the sphere with $n$ vertices can be transformed into each other by $O(n \log n)$ diagonal flips so that their labelings coincide.

### 5.1.3 Pseudo-triangulations

In Chapters 2, 3 and 4, when we consider a sequence of diagonal transformations in $N$-angulations, we preserve the simplicity of graphs, and this is an interesting point of these theorems. In this subsection, we focus on triangulations which might have multiple edges, called pseudo-triangulations, and then, we introduce results on diagonal flips in pseudo-triangulations.

A pseudo-triangulation $G$ on a closed surface $F^2$ is a graph (which might not be simple) embedded on $F^2$ so that each face of $G$ is bounded by a closed walk of length 3. Clearly, the set of triangulations on $F^2$ is the subset of the set of pseudo-triangulations on $F^2$. Moreover, a diagonal flip in pseudo-triangulations is defined as well as that in a usual triangulation, but it is not required to preserve the simplicity of graphs.
It is known that the theorem corresponding to Theorem 2.7 holds for pseudo-triangulations, but with no restriction on the number of vertices \([62]\). That is, \(N(F^2) = 1\) for all of closed surfaces \(F^2\). (In particular, a pseudo-triangulation with exactly one vertex is sometimes called a \textit{bouquet triangulation}.) Moreover, Negami and Watanabe \([67]\) proved the similar result for pseudo-triangulations with no loops as follows.

**Theorem 5.7 (Negami and Watanabe \([67]\))** For any closed surface \(F^2\), there exists a positive integer \(n(F^2)\) such that any two pseudo-triangulations \(G_1\) and \(G_2\) on \(F^2\) without loops can be transformed into each other by diagonal flips through those pseudo-triangulations if \(|V(G_1)| = |V(G_2)| \geq n(F^2)\).

Negami also established the similar theorem for “labeled” pseudo-triangulations. For the details of the result and of pseudo-triangulations, see \([60]\).

### 5.2 Other transformations in graphs on surfaces

In this section, we introduce several known results on local transformations in graphs on surfaces, and we show a result on one of those transformations.

#### 5.2.1 \(N\)-flips and \(P_2\)-flips in even triangulations

An \textit{even triangulation} is a triangulation \(G\) such that every vertex in \(G\) has even degree. Suppose that an even triangulation \(G\) has a hexagonal region \(v_1 v_2 v_3 v_4 v_5 v_6\) with diagonals \(v_1 v_3, v_3 v_6\) and \(v_4 v_6\) and no inner vertices. The \(N\)-flip of the path \(v_1 v_3 v_6 v_4\) is to replace the diagonals \(v_1 v_3, v_3 v_6\) and \(v_4 v_6\) with \(v_1 v_5, v_2 v_5\) and \(v_2 v_4\) in the hexagonal region as shown in the left hand of Figure 5.3. If the resulting graph is not simple, then we do not apply it.

Let \(G\) be an even triangulation and let \(v\) be a vertex of \(G\) with link \(v_1 v_2 \cdots v_k\). Put two vertices \(x\) and \(y\) on \(vv_1\) and join them to \(v_2\) and \(v_k\), and let \(G'\) be the resulting graph. The \(P_2\)-flip of \(\{x, y\}\) in \(G'\) is to move the inserted vertices \(x\) and \(y\) to the edge \(v_2v_5\) to join them to \(v_1\) and \(v_3\) as shown in the right hand of Figure 5.3. Similarly to the above, if the resulting graph is not simple, then we do not apply the operation.

![Figure 5.3: An N-flip and a P2-flip](image)

It is easy to see that both of \(N\)-flips and \(P_2\)-flips transform an even triangulation into an even triangulation. Nakamoto, Sakuma and Suzuki proved the following.

**Theorem 5.8 (Nakamoto et al. \([57]\))** Any two even triangulations on the sphere with the same number of vertices can be transformed into each other by \(N\)- and \(P_2\)-flips.
It is well known that every even triangulation $G$ on the sphere is 3-chromatic. In this case, $V(G)$ can be uniquely decomposed into three independent sets $V_r(G) \cup V_b(G) \cup V_y(G)$, where these classes are referred as red, blue and yellow vertices of $G$, respectively. Such a decomposition of $V(G)$ is called the tripartition of $V(G)$. The set $\{|V_r(G)|, |V_b(G)|, |V_y(G)|\}$ is called the tripartition size of $G$. As shown in Figure 5.3, the two transformations preserve the 3-chromaticity of a given even triangulation. Moreover, observe that an $N$-flip preserves a given tripartition size, but a $P_2$-flip does not. Hence, if two given even triangulations with the same number of vertices but distinct tripartition size cannot be transformed into each other only by $N$-flips. For even triangulations on the sphere, Nakamoto et al. [57] also proved that the opposite of the above is also true as follows.

**Theorem 5.9 (Nakamoto et al. [57])** Any two even triangulations on the sphere with the same tripartition size can be transformed into each other only by $N$-flips.

In general, an even triangulation on a closed surface other than the sphere might not be 3-colorable. For even triangulations on a closed surface other than the sphere, the equivalence can be described by a notion called a monodromy, where the detailed argument can be found in [37]. (A monodromy of even triangulations is a concept which slightly resembles in a cycle parity of quadrangulations.)

Now, we focus on the number of $N$-flips to transform an even triangulation on the sphere into another. A face subdivision of a graph $G$ on a closed surface, denoted by $FS(G)$, is the embedding obtained from $G$ by adding a new single vertex into each face of $G$ and joining it to all vertices on the corresponding boundary walk. Note that every even triangulation $G$ on the sphere is a face subdivision of an even-embedding $H$ on the sphere. Since if we remove one partite of the tripartite set of $V(G)$, then we always obtain a bipartite graph.

Let $G$ be an even triangulation on the sphere which is a face subdivision of a quadrangulation $Q$ on the sphere. In this case, as shown in Figure 5.4, a diagonal slide in $Q$ corresponds to an $N$-flip in $G$. Therefore, by Theorem 4.9, we can immediately obtain the following.

**Theorem 5.10** Let $Q$ and $Q'$ be quadrangulations on the sphere with the same bipartition size $(B,W)$, and let $T$ and $T'$ be even triangulations on the sphere which are face subdivisions of $Q$ and $Q'$, respectively. Then, $T$ and $T'$ can be transformed into each other by at most $10B + 16W - 64$ $N$-flips.

However, it seems to be difficult to extend Theorem 5.10 to face subdivisions of any even-embedding which is not a quadrangulation, since in general, there are two types of $N$-flips in even triangulations, that is, there are two local transformations in corresponding even-embeddings. One of them is called an edge slide shown in Figure 5.5, and the other is called an edge wipe shown in Figure 5.6.

Let $G$ be a graph on a closed surface and let $f_1, f_2, \ldots, f_{|F(G)|}$ be faces in $G$. A face size $x_i$ of $f_i$ is the length of $\partial f_i$. Moreover, a face size set $\{x_1, x_2, \ldots, x_{|F(G)|}\}$ is the set of all face sizes in $G$, where $x_i \leq x_{i+1}$ for any $i$. 


An edge slide is necessary for quadrangulations since every edge wipe requires a face which has the face size at least 6 to preserve the simplicity (in this case, the transformation is equivalent to a diagonal slide). Since an edge wipe changes face sizes of two faces sharing the corresponding edge in even-embeddings, the operation is needed when two given even embeddings have different face size sets. Moreover, since there are no result for edge slides and edge wipes in even-embeddings, we guess that we need to consider the two transformations to estimate the number of N-flips.

On the other hand, we have not yet found a pair of even triangulations on the sphere which requires $\Omega(n^2)$ N-flips to transform into each other, where $n$ is the number of vertices. (A function $f(n)$ is $\Omega(n)$ if $\lim_{n \to \infty} \frac{f(n)}{n} > 0$.) Thus, we conjecture the followings.

**Conjecture 5.11** Any two even triangulations on the sphere with the same tripartition size can be transformed into each other by $O(n)$ N-flips, where $n$ is the number of vertices.

**Conjecture 5.12** Any two even triangulations on the sphere with $n$ vertices can be transformed into each other by $O(n)$ N- and $P_2$-flips.
5.2.2 Signed diagonal flips

A signed triangulation is a pair $(T, \rho)$, where $T$ is a triangulation on a closed surface and $\rho : F(T) \rightarrow \{+1, -1\}$ is any sign function on the set $F(T)$ of faces of $T$. Let $(T, \rho)$ be a signed triangulation, and let $e$ be an edge in $T$. The signed diagonal flip of $e$ is only defined if the two faces $f_1$ and $f_2$ of $T$ sharing $e$ have the same sign. If $\rho(f_1) = \rho(f_2) = \alpha$, then the signed diagonal flip of $e$ in $(T, \rho)$ shown in Figure 5.7 results in the signed triangulation $(T', \rho')$ such that $T'$ is obtained from $T$ by flipping $e$ and $\rho'(f_1) = \rho'(f_2) = -\alpha$ and $\rho(f) = \rho'(f)$ for any other face $f$.

![Figure 5.7: A signed diagonal flip](image)

We introduce the relation between signed triangulations and 4-colorings of the graphs. Let $(G, \rho)$ be a signed triangulation of a triangulation $G$ on a closed surface $F^2$. We denote the subset of faces of $G$ incident to some vertex $v$ by $F_v$. Moreover, let $s_\rho(v)$ be the sum of the signs of faces $f$ incident to $v$, i.e.,

$$s_\rho(v) = \sum_{f \in F_v} \rho(f).$$

The signing $\rho$ is a Heawood signing if at every vertex $v$ of $G$, one has

$$s_\rho(v) \equiv 0 \mod 3.$$

Now, let us construction a Heawood signing by using a 4-coloring of the triangulation. Let $G$ be a triangulation on a closed surface $F^2$, and let $\gamma : V(G) \rightarrow F_4$ be a 4-coloring of
$G$ with values in the finite field $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$ with four elements. If $xy$ is an edge of $G$, define $\partial \gamma(xy) = \gamma(x) + \gamma(y)$, which is an element of $\mathbb{F}_4^* = \{1, \alpha, \alpha^2\}$. It is easy to see that the function $\partial \gamma$ is a Grünbaum coloring of $G$, i.e., a coloring of the edges with three colors such that the boundary of each face is colored by distinct three colors $1, \alpha$ and $\alpha^2$. Here, for each face $f$ of $G$, set $\rho(f) = +1$ if the colors $1, \alpha$ and $\alpha^2$ appear in a clockwise order on the edges of $f$, and $\rho(f) = -1$ otherwise. Using the relation $\alpha^3 = 1$ in $\mathbb{F}_4$, it is not difficult to check that the signing $\rho$ just defined is in fact a Heawood signing.

On the sphere, this construction can be reversed, as it can be shown that every Heawood signing of a triangulation $G$ is induced as above by a 4-coloring. However, for any other surface, the same fact does not hold in general. For more details on the relationship between 4-colorings, Heawood signings and Grünbaum colorings, see [22, 23, 30].

A signed maximal outerplane graph is a pair $(G, \rho)$, where $G$ is a signed maximal outerplane graph and $\rho : F(G) \rightarrow \{+1, -1\}$ is any sign function on the set $F(G)$ of inner faces of $G$. Two maximal outerplane graphs $T$ and $T'$ are said to be signable if there exist $(T, \rho)$ and $(T', \rho')$ such that they can be transformed into each other by a sequence of signed diagonal flips. (In this case, note that the outer faces of $T$ and $T'$ have no sign, that is, we cannot apply a signed diagonal flip to edges on the boundaries of the outer faces.) Eliahou [21] conjectured the following which is named the signed flips conjecture.

**Conjecture 5.13 (Signed flips conjecture [21])** Every two maximal outerplane graphs with the same number of vertices are signable.

Moreover, it is very interesting to note that if this conjecture is true, then we have the four color theorem.

**Theorem 5.14 (Eliahou [21])** The truth of the signed flips conjecture implies the four color theorem.

Therefore, we have a chance to give a new logical proof to the four color theorem by applying the argument for signed diagonal flips and by using no computer.

### 5.2.3 Simultaneous flips in triangulations

Wagner [78] proved that every triangulation on the sphere with $n + 3$ vertices can be transformed into the standard form $\Delta_n$, and $O(n^2)$ diagonal flips are needed in this algorithm. Komuro [38] improved Wagner’s algorithm and proved his algorithm requires $O(n)$ diagonal flips to transform two given triangulations on the sphere with $n$ vertices into each other. Moreover, since he also proved that there are two triangulations on the sphere with $n$ vertices which require at least $2n - 15$ diagonal flips to transform into each other. Hence, the linear order of Komuro’s estimation about the number of vertices is best possible. Now, how many will the number of flips decrease if we can flip many edges at a time?

Let $S$ be a set of edges in a triangulation $G$ on the sphere. The graph obtained from $G$ by flipping each edge in $S$ is denoted by $G(S)$. If $G(S)$ is a triangulation on the sphere, then $S$ is a flippable set in $G$, and $G$ is simultaneous flipped into $G(S)$ by $S$. This operation is called a simultaneous flip. Note that if a set $S$ is a flippable set, then a diagonal flip can
be applied to each edge in $S$. However, the converse does not always hold. For example, let $uxy$, $xyv$, $uab$ and $abv$ be four faces in a triangulation $G$ on the sphere, where $u \neq v$, $uv \notin E(G)$ and $\{x, y\} \cap \{a, b\} = \emptyset$. (see Figure 5.8). In this case, a diagonal flip can be applied each of $xy$ and $ab$ since $uv \notin E(G)$. However, as shown in Figure 5.8, the set $S = \{xy, ab\}$ is not a flippable set since the resulting graph has multiple edges.

![Figure 5.8: The set $S = \{xy, ab\}$ is not a flippable set.](image)

Recently, Bose et al. [12] proved many results on simultaneous flips in triangulations on the sphere. One of them is the following, which answers our previous question.

**Theorem 5.15 (Bose et al. [12])** Any two triangulations on the sphere with $n$ vertices can be transformed into each other by $O(\log n)$ simultaneous flips.

Bose et al. [12] also showed that the result is optimal since there are two triangulations on the sphere with $n$ vertices which require $\Omega(\log n)$ simultaneous flips to transform into each other. Moreover, they estimated the number of edges which can be contained in one flippable set.

### 5.2.4 Edge rotations in planar graphs

In this and next subsections, we always consider a graph obtained from a triangulation on the sphere by replacing several edges with dots edges (see Figure 5.9). A dots edge is called an **absent edge** and the other is called a **non-absent edge**. (In this case, $G$ may not be connected only by non-absent edges.) Following this, let triangulations mean graphs like $G$ in Figure 5.9, that is, a triangulation has at least one absent edge and at least one non-absent edge.

In a triangulation $G$ on a surface, as shown in Figure 5.10, replacing a non-absent edge $xu_i$ with $xu_j$ is called an **edge rotation**, but if at least one of the following two conditions is not satisfied, then we do not apply this transformation:

(i) This operation preserves the simpleness of $G$. (In this case, we regard two edges as multiple edges even if they are absent edges.)

(ii) $u_i$ and $u_j$ are not separated in the interior and the exterior by a cycle consisting of non-absent edges.

For edge rotations, the following proposition holds. The fact means that we can move an absent edge to any position by a sequence of edge rotations. (In [15], the authors proved
that any absent edge can be exchanged any non-absent edge by $O(n)$ edge rotations in a “geometric” triangulation on the plane. It is easy to see that their proof can be applied to a triangulation on any closed surface.)

**Proposition 5.16 (Cano et al. [15])** Let $G$ be a triangulation on a closed surface with $n$ vertices, and let $e$ and $e'$ be an absent edge and a non-absent edge in $G$, respectively. Then $e$ can be exchanged $e'$ by $O(n)$ edge rotations.

In fact, there exists the relation between a diagonal flip and an edge rotation. As shown in Figure 5.11, a diagonal flip in a triangulation $G$ is obtained from a sequence of edge rotations in $G$. For this operation, we need an absent edge on the boundary of the corresponding quadrilateral region. Hence, any diagonal flip in a triangulation $G$ with $|V(G)| = n$ can be obtained from $O(n)$ edge rotations by Proposition 5.16.

Therefore, we immediately obtain the following by results on the number of diagonal flips in triangulations in Chapter 4.

**Theorem 5.17** For any closed surface $F^2$, there exists a positive integer $n = N(F^2)$ such
Figure 5.11: A diagonal flip and a sequence of edge rotations

that any two triangulations on $F^2$ with $n$ vertices and $k$ edges can be transformed into each other by $O(n^2)$ edge rotations. 

So, we consider whether any two triangulations on a closed surface with $n$ vertices and $k$ edges can be transformed into each other by $O(n)$ edge rotations, and in this thesis, we prove the following.

**Theorem 5.18** Any two triangulations on the sphere with $n \geq 4$ vertices and $k$ edges can be transformed into each other by at most $32n - 106$ edge rotations.

Moreover, we boldly conjecture the following. (Note that $N(S_0) = 4$ in the following by Theorem 5.18.)

**Conjecture 5.19** For any closed surface $F^2$, there exists an integer $N(F^2)$ such that any two triangulations on $F^2$ with $n \geq N(F^2)$ vertices and $k$ edges can be transformed into each other by at most $O(n)$ edge rotations.

It is well-known that a triangulation on the sphere with $n$ vertices and no absent edge can be obtained from a simple graph $G$ on the sphere with the same number of vertices by joining $u$ and $v$ such that $u, v \in \partial f$ and $uv \notin E(G)$ for any face $f$ bounded by a closed walk of length at least four, preserving the planarity and simpleness of $G$. Hence, since those additional edges can be regarded as absent edges, the following is equivalent to Theorem 5.18.

**Theorem 5.20** Any two simple graphs on the sphere with $n \geq 4$ vertices and $k$ edges can be transformed into each other by at most $32n - 106$ edge rotations.

Note that if two triangulations on the sphere with $n = 3$ vertices are given, then we do not have to apply edge rotations since there are only two kinds of graphs shown in Figure 5.12. Therefore, we only consider the condition $n \geq 4$. 

99
5.2.5 Proof of Theorem 5.18

In this subsection, we shall prove Theorem 5.18. In the first, we shall prove the following lemma. In triangulations on the sphere, the degree of a vertex \( v \) means the total number of non-absent edges and absent edges incident to \( v \), is denoted by \( \text{deg}(v) \), and let \( N(v) \) be the neighbor set of \( v \) (\( N(v) \) includes vertices which are adjacent to \( v \) by absent edges). Moreover, for triangulations on the sphere without absent edges, the standard form \( \Delta_{n-3} \) is a triangulation without absent edges as shown in Figure 5.13.

![Figure 5.12: Graphs with \( n = 3 \)](image)

![Figure 5.13: The standard form \( \Delta_{n-3} \) (\( \text{deg}(y) = \text{deg}(z) = n - 1 \))](image)

Let \( uv \) be a non-absent edge and \( uw \) be an absent edge in a triangulation, and we consider to flip from \( uv \) to \( uw \) by an edge rotation. In this case, we denote \( uv \rightarrow uw \). Similarly, a diagonal flip from \( uv \) to \( wx \) is denoted by \( uv \rightarrow_d wx \) in triangulations (without absent edges).

**Lemma 5.21** Let \( G \) be a triangulation on the sphere with \( n \geq 4 \) vertices and \( k \) edges. Then \( G \) can be transformed into a triangulation obtained from \( \Delta_{n-3} \) by replacing \( 3n - 6 - k \) edges of \( G \) with absent edges (for example, see Figure 5.14) by at most \( 9n - 35 \) edge rotations.

**Proof.** Let \( G \) be a triangulation on the sphere with \( n \geq 4 \) vertices and \( k = 3n - 7 \) edges. Then \( k = 3n - 7 \) means that the number of absent edges is exactly one. (In the proof of this lemma, we may suppose \( k = 3n - 7 \) because this case requires the most number of edge rotations.) We first label a face as \( xyz \) such that \( xz \) is an absent edge. (This face corresponds to an outer face in Figure 5.13.) For the proof of this lemma, we define a constant \( d_G(y, z) = 3\text{deg}(y) + \text{deg}(z) \) which is used in the proof of Lemma 2 in [38].

Suppose that \( \text{deg}(x) \geq 4 \) and let \( z, v_1, v_2, ..., v_j, y \) be the neighbor of \( x \) lying around \( x \) in this order. If a non-absent edge \( v_2z \) doesn’t exist, we apply \( xv_1 \rightarrow xz \rightarrow v_2z \) which is
Figure 5.14: A triangulation obtained from $\Delta_{n-3}$ by removing several edges

equivalent to $xv_1 \to_d v_2z$. Otherwise, since $v_1y$ doesn’t exist, we apply $v_1z \to v_1y$ which is equivalent to $xz \to_d yv_1$. After this, we label $v_1$ as $x$.

In both cases, the operations increase $d_G(y,z)$ at least one by at most two edge rotations (corresponding to at most one diagonal flip), and $xz$ is always an absent edge in the end of the operations.

Next, we suppose $\deg(x) = 3$ and let $x_1$ be a unique common neighbor of $x, y$ and $z$. Before we apply the following operation, we apply $x_1z \to xz$. Suppose $\deg(x_1) \geq 5$ and let $z, u_1, u_2, \ldots, u_l, y$ be the neighbor of $x_1$ lying around $x_1$ in this order. If a non-absent edge $u_2z$ doesn’t exist, we apply $x_1u_1 \to x_1z \to u_2z$ which is equivalent to $x_1u_1 \to_d v_2z$.

Otherwise, since $u_1y$ doesn’t exist, we apply $xx_1 \to xu_1 \to yu_1$ and $u_1z \to u_1x$ which is equivalent to $x_1z \to_d xu_1$ and $xx_1 \to_d yu_1$. After this, we label $u_1$ as $x_1$.

In both cases, the operations increase $d_G(y,z)$ at least one by at most three edge rotations (corresponding to at most two diagonal flips), and $x_1z$ is always an absent edge in the end of the operations.

Moreover, if $\deg(x_1) = 3$, then we have already obtained the form like Figure 5.14, and if $\deg(x_1) = 4$, then there exist a common neighbor $x_2$ of $x_1, y$ and $z$. Then, we carry out the same operations inside the triangle $x_2yz$ as we did for the triangle $x_1yz$.

By repeating above operations until we have $\deg(x_{n-3}) = 3$, we can obtain $d_G(y,z) = 4n - 4$, that is, we have the form like Figure 5.14 since $\deg(y) = \deg(z) = n - 1$.

Now, we shall calculate the total number of edge rotations in this proof. In [38], Komuro proved that any triangulation without absent edges can be transformed into $\Delta_{n-3}$ by at most $4n - 4 - (3\deg(y) + \deg(z))$ diagonal flips by using the method similarly to this proof. In the above operations, each diagonal flip is obtained from at most two edge rotations, and $\deg(y) \geq 3$ and $\deg(z) \geq 3$ in the first condition. Moreover, we applied $n - 3$ edge rotations $xz \to x_1z \to \cdots \to x_{n-3}z$. Then, we have

$$2\{4n - 4 - (3 \times 3 + 3)\} + (n - 3) = 9n - 35.$$ 

Therefore, this lemma holds. ■

Now, we shall prove Theorem 5.18.
Proof of Theorem 5.18. Let $G$ be a triangulation with $n \geq 4$ vertices and $k$ edges. Then we shall prove that $G$ can be transformed into the standard form shown in Figure 5.15 depending on $k$ by at most $16n - 53$ edge rotations.

Figure 5.15: These are the standard forms, where non-absent edges are put on $xy, x_1y, \ldots, x_{n-3}y, xx_1, x_1x_2, \ldots, x_{n-4}x_{n-3}, zz, zz_1, \ldots, zz_{n-3}$ in this order corresponding to $k$.

By Lemma 5.21, we can transform $G$ into the form as shown in Figure 5.14 by at most $9n - 35$ edge rotations. Hence, we shall show that we can move all non-absent edges of $G$ by at most $7n - 18$ edge rotations to transform $G$ into the standard form. If $k = 3n - 7$, then we have the standard form by exactly one edge rotation since $x_{n-3}z$ is an absent edge after applying Lemma 5.21. Thus, we may suppose $k < 3n - 7$.

We first suppose that $i$ is the smallest number such that $yx_i$ ($i = 0, 1, \ldots, n - 3$ and $x_0 = x$) is an absent edge and $yx_j$ is a non-absent edge for any $j < i$. If $k < i$ or $i$ doesn’t exist (that is, $n - 2 \leq k$), go to after Step 3.

**Step 1.** If a non-absent edge $yx_l$ ($i < l \leq n - 3$) (or $yz$) exists, there exists $l$ such that we obtain a non-absent edge $yx_i$ by $yx_l \rightarrow yx_i$. Moreover, if $yz$ is a non-absent edge and $yx_l$ is an absent edge for any $l$ ($i < l \leq n - 3$), then we obtain a non-absent edge $yx_i$ by $yz \rightarrow yx_i$.

**Step 2.** If a non-absent edge $zx_h$ ($0 \leq h \leq n - 3$) exists (Step 1 doesn’t arise), there exists $h$ such that we obtain a non-absent edge $yx_i$ by $zx_h \rightarrow yz \rightarrow yx_i$.

**Step 3.** If a non-absent edge $x_mx_{m+1}$ ($0 \leq m \leq n - 4$) exists (Steps 1, 2 doesn’t arise), we obtain a non-absent edge $yx_i$ by $x_mx_{m+1} \rightarrow zx_m \rightarrow yz \rightarrow yx_i$.
By applying Steps 1, 2 and 3, we can obtain the standard form of $G$ if $k \leq n - 2$. Hence, we second suppose that $i$ is the smallest number such that $x_ix_{i+1}$ is an absent edge for $i$ ($i = 0, 1, \ldots, n - 4$) and $x_jx_{j+1}$ is a non-absent edge for any $j < i$. and $n - 2 < k \leq 2n - 5$. Moreover, we may suppose that we have already applied operations of Steps 1, 2 and 3 to $G$, that is, $yx_r$ is a non-absent edge for any $r$. Then, if $k = (n - 2) < i$ or $i$ doesn’t exist (that is, $2n - 5 \leq k$), go to after Step 5.

**Step 4.** If a non-absent edge $zx_h$ ($0 \leq h \leq n - 3$) exists, there exists $h$ such that we obtain a non-absent edge $x_ix_{i+1}$ by $zx_h \rightarrow zx_i \rightarrow x_ix_{i+1}$. (If $h = i$, then we can obtain a non-absent edge $x_ix_{i+1}$ by at most one edge rotation.)

**Step 5.** If a non-absent edge $x_ix_{i+1}$ is a non-absent edge for $i < t$ (Step 4 doesn’t arise), there exists $t$ such that we obtain a non-absent edge $x_ix_{i+1}$ by $x_ix_{i+1} \rightarrow zx_t \rightarrow zx_i \rightarrow x_ix_{i+1}$.

By applying Steps 4 and 5, we can obtain the standard form of $G$ since $n - 2 < k \leq 2n - 5$. Hence, we second suppose that $i$ is the smallest number such that $zx_i$ is an absent edge for $i$ ($i = 0, 1, \ldots, n - 3$) and $zx_j$ is a non-absent edge for any $j < i$. and $2n - 5 < k \leq 3n - 8$. Moreover, we may suppose that we have already applied operations of Steps 1 to 5 to $G$ in this order, that is, $yx_r$ and $x_sxs_{s+1}$ are non-absent edges for any $r, s$. Then, if $k = (2n - 5) < i$ or $i$ doesn’t exist (that is, $G$ is already isomorphic to the standard form), go to after Step 6.

**Step 6.** If a non-absent edge $zx_h$ ($i < h \leq n - 3$) exists, there exists $h$ such that we obtain a non-absent edge $zx_i$ by $zx_h \rightarrow zx_i$.

By repeating Steps 1 to 6 in this order, it is not difficult to check that $G$ can be transformed into the standard form by edge rotations. Here, we shall calculate the total number of edge rotations of the above operations. In Steps 1 to 5, we can obtain each non-absent edge by at most three edge rotations, and in Step 6, we can obtain it by exactly one edge rotation. Hence, we have

$$3(2n - 5) + (n - 3) = 7n - 18.$$  

Therefore, since we have $16n - 53$ by adding $9n - 35$ to the above number, the theorem holds. ■

### 5.3 Flips in geometric setting

Throughout this section, the graphs are straight-line planar embeddings where vertices are points in the plane and edges are straight-line segments (we sometimes call a such graph a geometric graph). In this section, we introduce results on flipping edges in geometric graphs.
5.3.1 Definitions and general results

Let $P$ be an $n$ points set arranged on the plane in general position, that is, no three points of $P$ lie on a straight line. (A point set considered in this section is always fixed on the plane. So we do not usually add “on the plane” to express it. Moreover, throughout this section, $P$ always implies an $n$ points set arranged on the plane in general position.) Let $\text{Conv}(P)$ denote the convex hull of $P$. A triangulation $T$ on $P$ is a 2-connected plane graph $G$ with $V(G) = P$ such that each edge is a straight segment, that the outer cycle of $G$ coincides with $\text{Conv}(P)$, and that each finite face is bounded by a 3-cycle. We say that an edge $e$ of $T$ can be flipped (simply, $e$ is said to be flippable in this section) if $e$ is shared by two inner faces $f_1$ and $f_2$ such that $C = f_1 \cup f_2$ is a convex quadrilateral as shown in the left hand of Figure 5.16, where the operation is called an edge flip. Moreover, an edge $e'$ is non-flippable if $e'$ is shared by two inner faces $f_1$ and $f_2$ such that $C = f_1 \cup f_2$ is not a convex quadrilateral as shown in the right hand of Figure 5.16, (Note that we do not flip each edge on $\text{Conv}(P)$.)

Lawson [44] proved that any two triangulations on $P$ can be transformed into each other by edge flips. Lawson’s proof also implies that $O(n^2)$ edge flips are sufficient to transform a triangulation on $P$ into another. On the other hand, Hurtado et al. [31] proved that the quadratic upper bound is tight, that is, they proved the following.

**Theorem 5.22 (Hurtado et al. [31])** There exists a pair of triangulations on $P$ which requires $\Omega(n^2)$ edge flips to transform one into the other.

This result differs from the topological case of planar triangulations since the number of diagonal flips is $O(n)$ in the case (cf. Theorem 4.4). In the end of this subsection, we introduce the proof of Theorem 5.22.

**Proof of Theorem 5.22.** Consider the $2n$ points set $Q$ with $\text{Conv}(Q) = p_1p_2q_nq_1$, and \{${p_1, p_2, \ldots, p_n}$\} lie on a convex curve and \{${q_1, q_2, \ldots, q_n}$\} lie on a convex curve. Next, we add lines $p_ip_{i+1}$ and $q_iq_{i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$, and add 4 edges $p_1p_n, p_nq_n, q_nq_1$ and $q_1p_1$. Then, we add edges into two $n$-gons $p_1p_2\ldots p_n$ and $q_1q_2\ldots q_n$ until every inner face in the two $n$-gons is a triangle. Let $H$ and $H'$ be resulting graphs obtained from $Q$ by the above process, that is, $H$ and $H'$ are the same graph now. Finally, we obtain two triangulations $T$ and $T'$ on $Q$ from $H$ and $H'$ by adding edges into the $2n$-gons so that there are edges $q_ip_i$ and $p_nq_i$ for all $i$ in $T$ and there are edges $p_1q_i$ and $q_np_i$ for all $i$ in
Figure 5.17: A pair of triangulations on $Q$ which requires $\Omega(n^2)$ edge flips ($n = 8$)

$T'$, respectively (for example, see Figure 5.17 in which we omit edges in four $n$-gons for the simplicity).

Now, we assign a code to faces in the $2n$-gons of $T$ and $T'$ as follows: Each triangle in the $2n$-gons has either two vertices in $\{p_1, p_2, \ldots, p_n\}$ or two vertices in $\{q_1, q_2, \ldots, q_n\}$. In the former case, assign a 1 to a triangle, and in the latter case, assign a 0 to a triangle. For example, in Figure 5.17, the left triangulation has a code 11111110000000 and the right one has a code 00000001111111.

By the above constructions, note that $T$ has a code $11 \ldots 100 \ldots 0$ and $T'$ has a code $00 \ldots 011 \ldots 1$. Observe that we can only apply an edge flip to an edge shared by two faces assigned 1 and 0. Moreover, after applying an edge flip to the edge, only two corresponding codes are switched. Therefore, it is not difficult to see that we need $(n-1)^2$ edge flips to transform the code $11 \ldots 100 \ldots 0$ into $00 \ldots 011 \ldots 1$.

### 5.3.2 Delaunay flips

Let $T$ be a triangulation on $P$. A triangle face $f$ of $T$ is called a Delaunay triangle if the circumcircle of $f$ (the circle passing through the vertices of $f$) contains no element of $P$ in its interior. A triangulation $T$ is a Delaunay triangulation denoted by $DT(P)$ if all of its triangle faces are Delaunay triangles. Moreover, it is known that when $P$ contains no four co-circular points, the Delaunay triangulation of $P$ is unique. Let $e$ be a flippable edge in a triangulation $T$ and let $Q_e = f_1 \cup f_2$ be a convex quadrilateral, where $f_1$ and $f_2$ are two faces sharing $e$. An edge flip of $e$ is said to be a Delaunay flip if there is a circle passing through the endvertices of $e$ which contains $Q_e$ in its interior (see Figure 5.18).

A Delaunay flip is introduced by Lawson [43], and it is known that if no Delaunay flip can be applied to a triangulation $T$ on $P$, then $T$ is the Delaunay triangulation on $P$ [43]. Moreover, Lawson also proved the following.
**Theorem 5.23** (Lawson [43, 44]) *Any triangulation on P can be transformed into the Delaunay triangulation on P by O(n²) Delaunay flips.*

By the above result, we can immediately see that any two triangulations on the same point set P can be transformed into each other by O(n²) edge flips through the Delaunay triangulation on P. Moreover, since there are many results and applications on Delaunay triangulations, the reader should refer to two books [10] and [25] for the details.

### 5.3.3 k-triangulations

A *k*-triangulation of a convex *n*-gon is a maximal set of edges such that no *k* + 1 of them mutually cross.

**Examples.** Every 1-triangulation (that is, there is no cross) is a maximal outerplane geometric graph. Moreover, if *n* ≤ 2* k* + 1, then the complete graph *K*ₙ of a convex *n*-gon does not have *k* + 1 mutually intersecting edges, and hence, it is the unique *k*-triangulation of the *n*-gon. So, the first non-trivial case is *n* = 2* k* + 2. Figure 5.19 shows a 2-triangulation of a convex 6-gon.

![Figure 5.19: A 2-triangulation of a convex 6-gon](image)

A *k*-triangulation is introduced by Capoyleas and Pach [16], who proved that any *k*-triangulation of a convex *n*-gon cannot have more than *k*(2*n* − 2* k* − 1) edges. Moreover, Nakamigawa [51] proved that all *k*-triangulations of a convex *n*-gon have the same number of edges which is equal to *k*(2*n* − 2* k* − 1) for *n* ≥ 2* k* + 1, and hence, the bound in [16] is best possible. The proof uses the concept of flips between *k*-triangulations. A flip
creates one $k$-triangulation from another one by removing and inserting a single edge (for example, see Figure 5.20).

![Figure 5.20: A flip in 2-triangulations of a convex 6-gon](image)

Nakamigawa also proved the following theorem. Note that since all $k$-triangulations have the same number of edges for $n \geq 2k + 1$, we do not need to consider whether the numbers of edges of two given $k$-triangulations are the same or not. (For more details of $k$-triangulations, see [69].)

**Theorem 5.24 (Nakamigawa [51])**  
For any positive integer $n \geq 2k + 1$, any two $k$-triangulations of a convex $n$-gon can be transformed into each other by flips through $k$-triangulations.

### 5.4 Edge operations in abstract graphs

In the final section, we introduce the research on edge operations in abstract graphs. Moreover, we describe known results on a concept which is similar to the transition diagram, called a distance graph.

#### 5.4.1 Distance between graphs by edge operations

Let $G$ and $H$ be two graphs with the same number of vertices, and let $\psi$ be an edge operation which is an operation of graphs preserving the number of vertices and the simplicity of graphs. The *distance* $\delta_\psi(G, H)$ between $G$ and $H$ is defined as the minimum value of $k$ such that $G$ can be transformed into $H$ by applying $\psi$ $k$ times, if such a $k$ exists; otherwise the distance is defined to be $\infty$. In this case, if we deal with every two graphs with the same number of vertices, then we simply denote the distance by $\delta_\psi$.

In this subsection, we consider the following three edge operations (note that the following transformations are different from transformations on surfaces with the same names which are defined so far):

An *edge move*, denoted by $EM$, is to replace an edge $e = uv$ with another edge $e' = u'v'$ (all of the four vertices might not be distinct), and we call the associate distance $\delta_{EM}$ the *edge move distance*. An *edge rotation*, denoted by $ER$, is to replace an edge $e = xy$ with another edge $e' = xz$, where $y \neq z$, and we call the associate distance $\delta_{ER}$ the *edge rotation distance*. An *edge slide*, denoted by $ES$, is to replace an edge $e = xy$ with another edge $e' = xz$, where $y \neq z$ and $y$ and $z$ are adjacent, and we call the associate
distance \( \delta_{ES} \) the *edge slide distance*. That is, the edge slide is a special type of the edge rotation. For the illustration, see Figure 5.21.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.21}
\caption{An edge move, an edge rotation and an edge slide}
\end{figure}

The edge rotation distance was introduced by Chartrand et al. [19], and the edge slide distance was defined by Johnson [33]. Though the name is from [9], the edge move distance was first defined by Baláž et al. [7] and Johnson [34].

It is easy to see that for all graphs \( G \) and \( H \):

\[
\delta_{EM}(G,H) \leq \delta_{ER}(G,H) \leq \delta_{ES}(G,H).
\]

Moreover, it has also been proved that (cf. [19]):

\[
\delta_{ER}(G,H) \leq 2\delta_{EM}(G,H).
\]

For any two graphs, Goddard and Swart [26] proved two simple bounds as follows. The *degree sequence* of a graph \( G \) is a monotonic increasing sequence of the degrees of the vertices in \( V(G) \).

**Theorem 5.25 (Goddard and Swart [26])** Let \( G \) and \( H \) be two graphs with \( n \) vertices and \( p(\binom{n}{2}) \) edges. Then

\[
\delta_{EM}(G,H) \leq p(1 - p)\left(\frac{n}{2}\right).
\]

**Theorem 5.26 (Goddard and Swart [26])** Let \( G \) be a graph with the degree sequence \((d_1 \leq d_2 \leq \cdots \leq d_n)\) and let \( H \) be a graph with the degree sequence \((t_1 \leq t_2 \leq \cdots \leq t_n)\). Then

\[
\delta_{ER}(G,H) \geq \frac{1}{2} \sum_{i=1}^{n} |d_i - t_i|.
\]

By the above results, we have the upper and lower bound of \( \delta_{ER} \). In general, when any two graphs are given, it is difficult to find the best bound of the distances \( \delta_{EM}, \delta_{ER} \) and \( \delta_{ES} \). However, if one of them is a tree and the other is a path or a star, then the distances are completely determined as follows.

**Proposition 5.27 (Benadé et al. [9], Zelinka [79, 80])** Let \( P_n \) and \( S_n \) be a path with \( n \) vertices and a star with \( n \) vertices, respectively. Then, for any tree \( T \) with \( n \) vertices,

(i) \( \delta_{EM}(T, S_n) = \delta_{ER}(T, S_n) = \delta_{ES}(T, S_n) = n - 1 - \Delta(T) \);

(ii) \( \delta_{ES}(T, P_n) = n - 1 - \text{diam}(T) \);

(iii) \( \delta_{ER}(T, P_n) = V_1(T) - 2 \);

where \( V_1(T) \) denotes the number of 1-vertices in \( T \).
Moreover, Goddard and Swart [26] showed that for any two trees $T$ and $T'$ with $n$ vertices, $\delta_{ER}(T, T') \leq n - 3$ and for any tree $U$ with $n$ vertices, there exists a tree $U'$ with $n$ vertices such that $\delta_{ER}(U, U') \geq n - o(n)$. (A function $f(n)$ is $o(n)$ if $\lim_{n \to \infty} \frac{f(n)}{n} = 0$.) Therefore, for any two trees $T$ and $T'$ with $n$ vertices, the upper bound $n - 3$ is best possible.

5.4.2 Distance graphs

In this subsection, for edge operations introduced in the previous subsection, we consider the distance graph which is a similar concept to the transition diagram for $N$-angulations. (The distance graph is first introduced in [18].) The edge move distance graph $D_M$ which consists of graphs with the same number of vertices such that two graphs $G$ and $H$ in $D_M$ are adjacent if and only if $\delta_{EM}(G, H) = 1$. Similarly, we define the edge rotation distance graph $D_R$ and the edge slide distance graph $D_S$ as defined above.

For the diameter of distance graphs, we have introduced several results as described in the previous subsection. Now, we next consider a problem: Which graphs can be distance graphs?

For the problem, there are many results. In particular, Chartrand et al. [18] proved that every graph can be an edge slide distance graph and conjectured that all graphs can be edge rotation distance graphs. Moreover, they also showed that several classes of graphs can be edge rotation distance graphs as follows.

Theorem 5.28 (Chartrand et al. [18]) Complete graphs, trees and cycles are edge rotation distance graphs.

In addition, Jarrett [32] showed that all wheels and complete bipartite graphs are also edge rotation distance graphs. However, no graph has been found which cannot be an edge rotation distance graph.

For edge move distance graphs, as far as we know, there are no result since the edge operation is very flexible. In fact, only when an edge move using four distinct vertices is allowed, Chartrand et al. [17] proved that several classes of graphs can be edge move distance graphs. (In [17], the corresponding edge operation is called an edge jump.) However, similarly to edge rotation distance graphs, a graph has not yet been found, which cannot be an edge move distance graph.

5.4.3 Equivalence for graphs with the same degree sequence

Let $G$ be a simple graph and let $e = xy$ and $f = uv$ be two edges of $G$. A 2-switch is to replace $e$ and $f$ with $xu$ and $yv$ (or $xv$ and $yu$), respectively, as shown in Figure 5.22. If this operation breaks the simplicity of $G$, then we do not apply it.

Clearly, a 2-switch preserves the degree sequence of $G$. Hence, if two graphs $G$ and $H$ can be transformed into each other by 2-switches, then $G$ and $H$ have the same degree sequence. In fact, by using 2-switches, Hakimi [28] and Havel [29] independently characterized those graphs as follows.
Figure 5.22: A 2-switch

**Theorem 5.29 (Hakimi [28], Havel [29])** Two simple graphs $G$ and $H$ have the same degree sequence if and only if $G$ can be transformed into $H$ by a sequence of 2-switches.

An application of such a result to analyze the network design can be found in [11]. Moreover, there are several results on a few variations of transformations related to the degree sequence, see [27].
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Index

1-sided, 23
1-subdivided triangulation, 60
2-cell embedded, 23
2-cell embedding, 23
2-sided, 23
2-switch, 109
$N_k$, 23
$S_g$, 23

A
absent edge, 97
abstract graph, 23
adjacent, 14

B
bipartite, 17
bipartition size, 17
boundary cycle, 23
boundary walk, 23

C
$k$-chromatic, 17
closed, 15
closed curve, 23
closed surface, 22
$k$-colorable, 17
$k$-coloring, 17
complete, 17
complete bipartite graph, 17
$k$-connected, 15
connected, 15
connected sum, 22
contractible, 23, 30
contracting, 15
contraction, 30
convex regular polyhedrons, 19
d-covered, 88
cut set, 16
cut vertex, 16

$k$-cycle, 15
cycle, 15
even cycle, 15
odd cycle, 15

D
degree, 14
degree sequence, 108
Delaunay flip, 105
Delaunay triangle, 105
Delaunay triangulation, 105
diagonal flip, 26
diagonal rotation, 32
diagonal slide, 32
diagonal transformation, 25
diameter, 17
disconnected, 15
distance, 17, 107
dual graph, 21

E
degree, 14
degree sequence, 108
diagonal flip, 26
diagonal rotation, 32
diagonal slide, 32
diagonal transformation, 25
diameter, 17
disconnected, 15
distance, 17, 107
dual graph, 21
essential, 23
Euler characteristic, 24
Euler’s formula, 24
even-embedding, 21
even-embedding, 23
even triangulation, 92

F
face, 18, 23
face size, 93
face size set, 93
face subdivision, 93
flippable, 44
flippable set, 96
forest, 16
Four color problem, 22
Four color theorem, 22
d-frozen, 88

G
greenometric, 103
greenath, 14
Grünbaum coloring, 96

H
Hamilton cycle, 15
Hamiltonian, 15
Heawood signing, 95
hexangulation, 25
homeomorphic, 23
homotopic, 23

I
incident, 14
induce, 15
irreducible, 30
isomorphic, 15
isomorphism, 15

J
join, 14
Jordan Curve Theorem, 18

K
Kuratowski’s Theorem, 20

L
labeling, 90
length, 15
link, 21
loop, 14

M
maximal outerplane graph, 77
maximum degree, 14
minimal, 28
minimum degree, 14
Möbius band, 22
Möbius wheel, 89
monodromy, 93
multiple edge, 14
multi-triangulation, 61

N
N-angulation, 25
neighborhood, 15
N-flip, 92
non-absent edge, 97
non-contractible, 23
non-flippable, 104
non-orientable, 22, 23
non-separating, 23

O
odd-embedding, 23
open 2-cell, 23
orientable, 22
outer region, 18

P
P₂-flip, 92
(u, v)-path, 15
k-path, 44
path, 15
pentangulation, 25
planar, 18
plane graph, 18
Platonic solids, 19
pseudo double wheel, 89
pseudo-minimal, 28
pseudo-triangulation, 91
Q
quadrangulation, 25

R
regular, 15
removing, 15

S
Schönhflies Theorem, 18
separating, 23
signable, 96
signed diagonal flip, 95
signed flips conjecture, 96
signed maximal outerplane graph, 96
signed triangulation, 95
simple, 14
simple closed curve, 18
simultaneous flip, 96
simultaneous flipped, 96
spanning, 16
spanning tree, 16
sphere graph, 18
standard form (triangulation), 26
standard form (quadrangulation), 34
standard form (pentangulation), 52
standard form (hexangulation), 41
star, 17
strongly equivalent, 90
subdivision, 20
subgraph, 15

transition diagram, 51
tree, 16
triangulation, 25
triangulation (geometric), 104
k-triangulation, 106
labeled triangulation, 90
tripartition, 93
tripartition size, 93
trivial, 23

U
up to homeomorphism, 24

V
vertex, 14